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A Study of Ecosystem Model for Survival of Aquatic Species

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Abstract— Oxygen is essential for all plants and animals to survive, whether they live on the land or in the water. Aquatic species rely on oxygen that is dissolved in the water (DO). If water is polluted because of discharge of pollutants or the water body acquires a high level of concentration of nutrients resulting excessive growth of algae (the process is called Eutrophication), due to which the level of dissolved oxygen decreases to support their lives. To analyze this ecological problem a conceptual and mathematical model is proposed. This model consists of set of nonlinear differential equations with boundary conditions. After obtaining the solution of these equations it is important to study about the stability of the solution by using stability theory.

Key words: Mathematical model, water pollution, dissolved oxygen (DO), algal bloom, Eutrophication.

I. INTRODUCTION

The water bodies near to residential areas or factories are most polluted because of constant discharge of wastes. The organic matters present in wastes are taken by small biological species (eg. bacterias) present in water and they convert them into inorganic matters using dissolved oxygen (DO) (hence level of DO decreases).A reason for decreased DO may be fertilizer runoff from farm fields and lawns. The same fertilizer which was meant to make land plants grow better now makes the aquatic plants do the same. If the weather becomes cloudy for several days, respiring plants will use much of the DO while failing to photosynthesize. When the increased numbers of aquatic plants eventually die, they support increasing amounts of bacteria which use large amounts of DO.

There is a food chain present in water body-

Bacteria (microorganism) -----> Protozoa--------->other biological species------> Fishes

This ecological problem has been studied since long. The first model of effect of organic pollutants on dissolved oxygen in a water body produced by Streeter Phelps (1925).Since then this model has been generalized by many researchers including Dobbins (1964), O Conner (1967), Beck and Young(1973) and the research is still going on.

It is important to mention here that these models have only linear differential equations. They have not considered the non-linear process that exists into water body.

There are several other aspects such as natural depletion rate of pollutants, crowding of population (they kill each other to survive) growth of oxygen from green plants like algae, macrophytes etc during photosynthesis have also not been taken into account in these studies but should be considered. It may also be noted here that the level of dissolved oxygen can be increased by pumping air into the water body such as in lakes and big ponds which are used for production of fish.

Therefore, these aspect needs to be considered in the modeling process.

II. STATISTICS OF DO IN WATER BODY

DO levels rise from morning through the afternoon as a result of photosynthesis, reaching a peak in late afternoon? Photosynthesis stops at night, but plants and animals continue to respire and consume oxygen. As a result, DO levels fall to a low point just before dawn? Dissolved oxygen levels may dip below 4 mg/l in such waters - the minimum amount needed to sustain warm water fish like bluegill, bass, and pike.

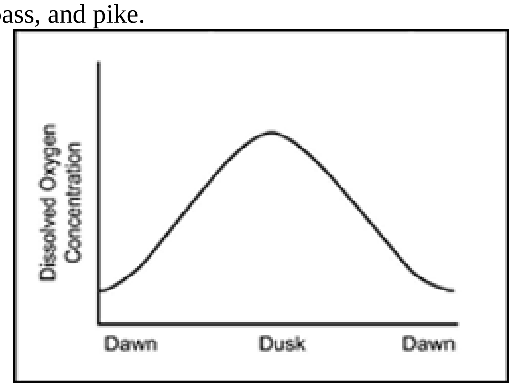

Fig. 1:

The oxygen content of natural waters varies with temperature, salinity, turbulence, photosynthetic activity of algae and plants, and atmospheric pressure. Primary sources of oxygen in water bodies include diffusion of atmospheric oxygen across the air-water interface and photosynthesis of aquatic plants

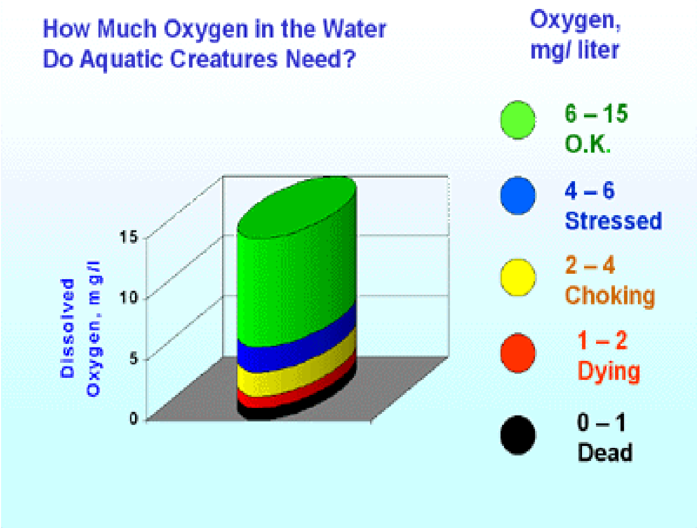

Fig. 2:

Thus, variations can occur seasonally as well as over 24-hour periods in response to temperature and biological activity. Dissolved Oxygen in a stream may vary from 0 mg/l to 18 mg/l. Readings above 18 mg/l are physically impossible Concentrations below 5 mg/l may adversely affect function and survival of biological communities, and below 2 mg/l can lead to death of most fishes.The amount of DO an aquatic organism needs depends upon its species, the temperature of the water, pollutants present, and the state of the organism itself (adult or young, active or dormant).

III. CONCEPTUAL MODEL

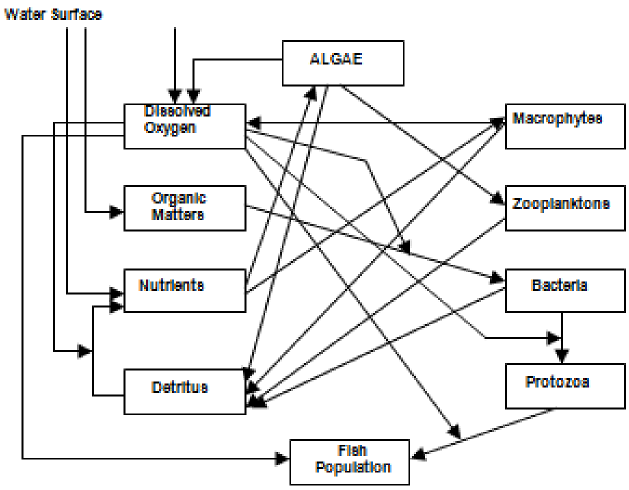

Fig. 3:

I. Mathematical Model:

Let us consider a water body where organic matters are discharged in the form of wastes.

It is assumed that these pollutants are part of a food chain consisting of bacteria, protozoa and other aquatic species using the dissolved oxygen into water body for various biochemical and biodegradation process.

Let us consider following notations

T – The cumulative concentration of organic matters (degradable pollutants).
B – The density of bacteria.
P – The density of protozoa.
N – The density of an aquatic population such as fish which depends wholly on protozoa.
C – The concentration of dissolved oxygen.
Q – The constant rate of cumulative discharge of organic pollutants in water body.
q – The growth rate of dissolved oxygen by various sources including atmospheric diffusion and pumping of air into water body.

A. System of nonlinear differential equations

$$\frac{dT}{dt} = Q - \alpha_0 T - \frac{K_1 TB}{K_{12} + K_{11} T}$$

$$\frac{dB}{dt} = \frac{\lambda_1 K_1 TB}{K_{12} + K_{11} T} - \alpha_1 B - \lambda_{10} B^2 - \frac{K_2 BP}{K_{21} + K_{22} B}$$

$$\frac{dP}{dt} = \frac{\lambda_2 K_2 BP}{K_{21} + K_{22} B} - \alpha_2 P - \lambda_{20} P^2 - K_3 PN \qquad (1.1)$$

$$\frac{dN}{dt} = \lambda_3 K_3 PN - \alpha_3 N - \lambda_{30} N^2$$

$$\frac{dC}{dt} = q - \alpha_4 C - \lambda_{12} \frac{K_1 TB}{K_{12} + K_{11} T} - \lambda_{23} \frac{K_2 BP}{K_{21} + K_{22} B} - \lambda_{34} K_3 PN - \lambda_{11}\alpha_1 B - \lambda_{22}\alpha_2 P - \lambda_{33}\alpha_3 N$$

Where T(0)>0 , B(0)>0 , P(0)>0 , N(0)>0 , C(0)>0

In this model all the coefficients are assumed positive. The constants $\lambda_{10}, \lambda_{20}, \lambda_{30}$ are crowding coefficients of bacteria, protozoa and aquatic population respectively.

In the following lines, we analyzed this model by using the qualitative theory of differential equations. It may be pointed out here that for feasibility of the above mathematical model, the growth rate of bacteria and protozoa should be positive.

Now we find the region of attraction Ω for model (1.1), the proof of which is outlined here in the following lemma

Feasibility of Model (1.1)

The region of attraction in the positive octant, is given by

$$\Omega : \left\{ 0 \le T \le \frac{Q}{\alpha_0}, 0 \le B \le R_B, 0 \le P \le R_P, 0 \le N \le R_N, 0 \le C \le \frac{q}{\alpha_4} \right\}$$

$$R_B = \frac{\lambda_1 K_1 Q}{\lambda_{10}(K_{11} Q + K_{12}\alpha_0)}, R_p = \frac{\lambda_1 \lambda_2 K_1 K_2 Q}{\lambda_{20}[K_{22}\lambda_1 K_1 Q + \lambda_{10} K_{21}(K_{11} Q + K_{12}\alpha_0)]}$$

where

$$R_N = \frac{\lambda_1 \lambda_2 \lambda_3 K_1 K_2 Q}{\lambda_{20}\lambda_{30}[K_{22}\lambda_1 K_1 Q + \lambda_{10} K_{21}(K_{11} Q + K_{12}\alpha_0)]}$$

Proof:

From the first equation of the model (1.1), we get

$$\frac{dT}{dt} \le Q - \alpha_0 T$$

$$\Rightarrow \frac{dT}{Q - \alpha_0 T} \le dt \quad .$$

Integrating both sides with respect to t,

$$\int \frac{dT}{Q - \alpha_0 T} \le \int dt \quad .$$

$$\Rightarrow -\frac{1}{\alpha_0} \log(Q - \alpha_0 T) \le \int dt + \log c_1 \quad .$$

$$\Rightarrow \log(Q - \alpha_0 T) \ge -\alpha_0 t - \alpha_0 \log c_1 \quad .$$

{multiplying by -α_0 both sides }

$$\Rightarrow \log(Q - \alpha_0 T) + \alpha_0 \log c_1 \ge -\alpha_0 t$$

Let

$$\log c_1^{\alpha_0} = k_1$$

$$\Rightarrow \log(Q - \alpha_0 T) k_1 \ge -\alpha_0 t$$

$$\Rightarrow (Q - \alpha_0 T) k_1 \ge e^{-\alpha_0 t}$$

$$\Rightarrow (Q - \alpha_0 T \geq \frac{e^{-\alpha_0 t}}{k_1}$$

$$\Rightarrow -\alpha_0 T \geq \frac{e^{-\alpha_0 t}}{k_1} - Q$$

On dividing the above equation by - α_0 we get,

$$T \leq \frac{e^{-\alpha_0 t}}{-k_1\alpha_0} + \frac{Q}{\alpha_0}$$

$$\Rightarrow T \leq -\frac{e^{-\alpha_0 t}}{k_1\alpha_0} + \frac{Q}{\alpha_0}$$

Since $k_1 > 0, \alpha_0 > 0$ and $e^{-\alpha_0 t} > 0$

Therefore, $\frac{e^{-\alpha_0 t}}{k_1\alpha_0} > 0$

Hence $T \leq \frac{Q}{\alpha_0}$

Since T be the cumulative concentration of organic matters which is positive

Hence $T \geq 0$

$\Rightarrow 0 \leq T \leq \frac{Q}{\alpha_0}$, This is the region of attraction for T

Now let's consider the second equation of the model (1.1), we get,

$$\Rightarrow \frac{dB}{dt} \leq -\frac{\lambda_1 k_1 TB}{k_{12} + k_{11}T} - \lambda_{10}B^2$$

$$= \frac{\lambda_1 k_1 B}{\frac{k_{12}}{T} + k_{11}} - \lambda_{10}B^2$$

$$\Rightarrow \frac{dB}{dt} \leq \frac{\lambda_1 k_1 B}{\frac{k_{12}\alpha_0}{Q} + k_{11}} - \lambda_{10}B^2$$

Since $\Rightarrow T \leq \frac{Q}{\alpha_0}$

$$\Rightarrow \frac{dB}{dt} \leq \frac{\lambda_1 k_1 BQ}{k_{12}\alpha_0 + k_{11}Q} - \lambda_{10}B^2$$

Let $\frac{\lambda_1 k_1 Q}{k_{12}\alpha_0 + k_{11}Q} = m_1$, a constant and $\lambda_{10} = m_2$

Hence the equation becomes,

$$\frac{dB}{dt} \leq m_1 B - m_2 B^2$$

This is a linear differential equation, we can solve it by variable separable method,

$$\Rightarrow \frac{dB}{m_1 B - m_2 B^2} \leq dt$$

$$\Rightarrow \frac{dB}{B(m_1 - m_2 B)} \leq dt$$

By using partial fraction, it can be written as,

$$\frac{dB}{m_1 B} + \frac{m_2}{m_1}\frac{dB}{(m_1 - m_2 B)} \le dt$$

Now integrating both sides,

$$\int \frac{dB}{m_1 B} + \frac{m_2}{m_1}\int \frac{dB}{(m_1 - m_2 B)} \le \int dt$$

$$\frac{1}{m_1}\log B + \frac{m_2}{m_1}\frac{\log(m_1 - m_2 B)}{-m_2} \le t + \log c_1$$

$$\frac{1}{m_1}\log B - \frac{1}{m_1}\log(m_1 - m_2 B) \le t + \log c_1$$

$$\frac{1}{m_1}\log \frac{B}{(m_1 - m_2 B)} \le t + \log c_1$$

$$\Rightarrow \log \frac{B}{(m_1 - m_2 B)} \le m_1 t + m_1 \log c_1$$

$$\Rightarrow \log \frac{B}{(m_1 - m_2 B)} - m_1 \log c_1 \le m_1 t$$

let $c_1^{k_1} = c_2$

$$\Rightarrow \log \frac{B}{(m_1 - m_2 B)}\frac{1}{c_2} \le m_1 t$$

$$\Rightarrow \frac{B}{(m_1 - m_2 B)}\frac{1}{c_2} \le e^{m_1 t}$$

$$\Rightarrow B \le (m_1 - m_2 B)c_2 e^{m_1 t} \quad \text{……………..(i)}$$

as we know B is the density of protozoa hence $B \ge 0$.
Consider the right hand side of above inequality,

$$m_2 > 0$$

$c_2 > 0$ (since $c_2 = c_1^{m_1}$ and c_1 is positive constant) , $e^{m_1 t} > 0$

$$\Rightarrow m_2 c_2 e^{m_1 t} > 0$$

hence the bracket part of equation (i) has to be positive

therefore, $m_1 - m_2 B \ge 0$

$$\Rightarrow B \le \frac{m_1}{m_2}$$

Now after substituting the value of m_1 and m_2, we get

$$\Rightarrow B \le \frac{\lambda_1 k_1 Q}{\lambda_{10}(k_{11}Q + k_{12}\alpha_0)}$$

$$\Rightarrow 0 \le B \le R_B$$

Hence the second result
Similarly from the third and fourth equation of the model (1.1), we get,

$\Rightarrow 0 \le P \le R_P$ Where $R_p = \dfrac{\lambda_1\lambda_2 K_1 K_2 Q}{\lambda_{20}[K_{22}\lambda_1 K_1 Q + \lambda_{10} K_{21}(K_{11}Q + K_{12}\alpha_0)]}$

and $0 \le N \le R_N$ where $R_N = \dfrac{\lambda_1\lambda_2\lambda_3 K_1 K_2 Q}{\lambda_{20}\lambda_{30}[K_{22}\lambda_1 K_1 Q + \lambda_{10} K_{21}(K_{11}Q + K_{12}\alpha_0)]}$

Now consider the last equation of the model,

$$\frac{dC}{dt} \leq q - \alpha_4 C$$

$$\frac{dC}{q - \alpha_4 C} \leq dt$$

After integrating both sides with respect to t, we get,

$$-\frac{1}{\alpha_4}\log(q - \alpha_4 C) \leq t + \log n_1$$

(n_1 is a constant)

Multiplying above inequality by $-\alpha_4$, we get,

$$\log(q - \alpha_4 C) \geq -\alpha_4 t - \alpha_4 \log n_1$$

$$\Rightarrow \log(q - \alpha_4 C)n_2 \geq -\alpha_4 t \quad \text{(let } n_1^{\alpha_4} = n_2\text{)}$$

$$\Rightarrow (q - \alpha_4 C) \geq \frac{e^{-\alpha_4 t}}{n_2}$$

$$\Rightarrow C \leq \frac{q}{\alpha_4} - \frac{e^{-\alpha_4 t}}{n_2 \alpha_4}$$

Here $\frac{e^{-\alpha_4 t}}{n_2 \alpha_4} > 0$ (sine $n_2 > 0, \alpha_4 > 0$) and $e^{-\alpha_4 t} > 0$

$$\Rightarrow C \leq \frac{q}{\alpha_4}$$

Since C is the concentration of dissolved oygen hence it is positive

$$\Rightarrow 0 \leq C \leq \frac{q}{\alpha_4}$$

Hence the lemma is proved.

IV. CONCLUSION

From the above analysis we get the region of attraction of each variable. Therefore the range of each parameter which is involved in the modeling process is obtained which is very useful in the further analysis of this system. This lemma helps to evaluate the equilibrium points of the system, their existence and stability. Further this model can be analyzed by considering two possibilities:-

1) When Q = 0 i.e. when organic matters (pollutants) are discharged instantaneously into a water body at t = 0 with the cumulativewe concentration $T_0 > 0$
2) When $Q \neq 0$ i.e. the discharge rate Q is a constant. By considering these cases we can analyze the stability of equilibrium points.

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