Transverse Shear Stresses for Simply Supported Beam Subjected to Various Loading Conditions

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Abstract— A new hyperbolic shear deformation theory for flexure of thick or deep beams, taking into account transverse shear deformation effects, is developed. The noteworthy feature of this theory is that the transverse shear stresses can be obtained directly from the use of constitutive relations with excellent accuracy, satisfying the shear stress free conditions on the top and bottom surfaces of the beam. Hence, the theory obviates the need of shear correction factor. In this paper we have studied transverse shear stresses for simply supported beams subjected to various loading. Results obtained are discussed critically with those of other theories.

Key words: Transverse Shear Stresses, Beam

I. INTRODUCTION

The beam and plate theories are the active areas of research since the historical time. The classical engineering theory of beam bending due to Bernoulli and Euler dates back to1705 and had its origin in the first mathematical model of nature of the resistance of a beam developed by Galileo Galilei in 1638. Saint Venant in 1856 presented the complete solution of the beam problems considering bending and shear stresses. The classical theory of plate bending had its origin in the pioneering work of Sophie Germain carried out in 1815. The theory reached maturity due to the well-known Kirchhoff hypothesis and the resolution of famous boundary conditions paradox by Kirchhoff in 1850.

Thick beams and plates, either isotropic or an isotropic, basically form two-and three dimensional problems of elasticity theory. Reduction of these problems to the corresponding one- and two-dimensional approximate problems for their analysis has always been the main objective of research workers. The shear deformations in beams and plates with the three dimensional nature of these problems further intensified the research interest in their accurate analysis. As a result, numerous refined theories of beams and plates have been formulated in last three decades which approximate the three dimensional solutions with reasonable accuracy.

I. NECESSITY OF REFINED THEORIES

The shear deformation effects are more pronounced in the thick beams when subjected to transverse loads than in the thin beams under similar loading. The shear deformation effects are more significant in the thick beams. These effects are neglected in Elementary Theory of Beam (ETB). In order to describe the correct bending behavior of thick beams including shear deformation effects and the associated cross sectional warping, shear deformation theories are required. This can be accomplished by selection of proper kinematic and constitutive models.

II. LITERATURE REVIEW

A refined theory containing the trigonometric or hyperbolic function in thickness coordinate, in the displacement field is termed here as new hyperbolic shear deformation theory (NHPSDT). The trigonometric functions involving thickness coordinate are associated with transverse shear deformation effects and the shear stress distribution through the thickness of the beam. This is another class of refined theories in which number of displacement variables in the simplest form can be same as those in FSDT.

Refined beam theories with the introduction of trigonometric function in terms of thickness coordinate in kinematical assumptions are introduced by Vlasov and Leont'ev [4], Stein [62]. It is interesting to note that in respect of plates similar trigonometric functions in thickness coordinates were used by Levy [63], Kil'chevskiy [2] and, as has been mentioned by Love [64], also by Kelvin and Tait.

The displacement field of the Vlasov and Leont'ev theory is as:

$$u(x,z) = zu_1(x) + u(x)\sin\left(\frac{2\pi z}{h}\right)$$

$$w(x,z) = w_1(x) + v_2(x)\cos\left(\frac{2\pi z}{h}\right)$$

The displacement field of the Stein's theory is as:

$$u(x,z) = zu_1(x) + u_2(x)\sin\left(\frac{\pi z}{h}\right)$$

$$w(x,z) = w_1(x) + v_2(x)\left(\frac{y}{b}\right)^2$$

In these displacement fields, the first right hand term represent the displacements when the sections are assumed to remain plane; the second terms are introduced to correct the inaccuracies due this assumption and that of zero transverse elongations.

However, in these theories shear stress free boundary conditions are not satisfied on the top and bottom surfaces of the beam. This deficiency is removed by Rao [65] in a refined theory developed for beams. The theory is as simple as first order shear deformation theory. However, it is variationally inconsistent. Ghugal and Shimpi [66] developed a variationally consistent refined trigonometric shear deformation theory for flexure and free vibration of thick isotropic beams. The displacement field of the theory in its simplest form is as:

$$u(x,z) = u_0(x) - z\frac{dw}{dx} + \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)\phi(x)$$

$$w(x,z) = w(x)$$

Further applications of this theory with general solution technique are given by Ghugal [67] and Ghugal and Waghe [68] for the flexural analysis of thick beams.

Sonawane and Ghugal [69] developed a variationally consistent refined trigonometric shear deformation theory for flexure and free vibration of thick isotropic beams. This theory takes into account transverse shear and normal flexibility effects. The displacement field of the theory in its simplest form is as:

$$u(x,z,t) = -z\frac{\partial w(x,t)}{\partial x} + \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)\phi(x,t)$$

$$w(x,z,t) = w(x,t) + \frac{h}{\pi}\cos\left(\frac{\pi z}{h}\right)\psi(x,t)$$

The number of displacement variables in this theory is three as compared to two in the FSDT. The theory satisfies the zero transverse shear stress conditions on the top and bottom surfaces of the beam and thus obviates the need of shear correction factor.

Ghugal and Sharma [70, 71], Sayyad and Ghugal [72] developed a variationally consistent refined hyperbolic shear deformation theory for flexure and free vibration of thick isotropic beam. This theory takes into account transverse shear deformations effects. The displacement field of the theory [70] in its simplest form is as:

$$u(x,z,t) = -z\frac{\partial w(x,t)}{\partial x} + \left[z\cosh\left(\frac{1}{2}\right) - h\sinh\left(\frac{z}{h}\right)\right]\phi(x,t)$$

$$w(x,z,t) = w(x,t)$$

Here, u is the axial displacement component in the x direction, and w is the transverse displacement in the z direction. The hyperbolic function in terms of thickness coordinate in the displacement u is associated with the transverse shear stress distribution through the thickness of beam and the function ϕ (x, t) is an unknown function associated with the shear slope.

Wang et al, [73] provided the relationship of solutions of shear deformable beams and plates with classical solutions. The relationships between Euler-Bernoulli beam theory, Timoshenko beam theory and Reddy-Bickford beam theory are presented.

The displacement field of the present refined beam theory can be expressed as follows:

$$u(x,z) = -z\frac{dw}{dx} + \left[\begin{array}{l} z\cosh\left(\frac{\alpha}{2}\right) - \\ \frac{h}{\alpha}\sinh\left(\frac{\alpha z}{h}\right)\end{array}\right]\phi(x)$$

$$w(x,z) = w(x)$$

III. Development of Theory

The beam under consideration as shown in Figure1 occupies in $0-x-y-z$ Cartesian coordinate system the region:

$$0 \le x \le L\ ; \quad 0 \le y \le b\ ; \quad -\frac{h}{2} \le z \le \frac{h}{2} \qquad (1)$$

where *x*, *y*, *z* are Cartesian coordinates, *L* and *b* are the length and width of beam in the x and *y* directions respectively, and *h* is the thickness of the beam in the z-direction. The beam is made up of homogeneous, linearly elastic isotropic material.

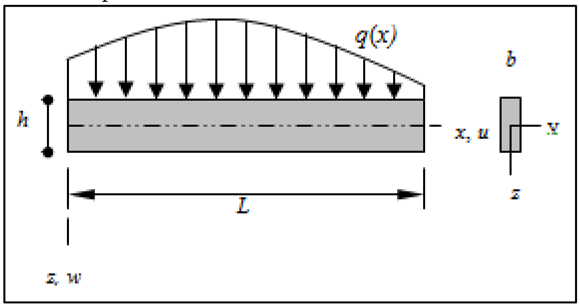


Fig. 1: Beam under bending in *x-z* plane

A. *The displacement field*

The displacement field of the present beam theory is of the form:

$$u(x,z) = -z\frac{dw}{dx} + \left[\begin{array}{l} z\cosh\left(\frac{\alpha}{2}\right) - \\ \frac{h}{\alpha}\sinh\left(\frac{\alpha z}{h}\right)\end{array}\right] \qquad (2)$$

$$w(x,z) = w(x)$$

where u the axial displacement in x direction and w is the transverse displacement in *z* direction of the beam. The sinusoidal function is assigned according to the shear stress distribution through the thickness of the beam. The function ϕ represents rotation of the beam at neutral axis, which is an unknown function to be determined. The normal and shear strains obtained within the framework of linear theory of elasticity using displacement field given by Eqn. (1) are as follows.

Shear strain: $\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{dw}{dx} = \cos\frac{\pi z}{h}\phi$ (3)

The stress-strain relationships used are as follows:

$$\tau_{zx} = G\gamma_{zx} \qquad (4)$$

B. *Governing equations and boundary conditions*

Using the expressions for strains and stresses (2) through (4) and using the principle of virtual work, variationally consistent governing differential equations and boundary conditions for the beam under consideration can be obtained. The principle of virtual work when applied to the beam leads to:

$$b\int_{x=0}^{x=L}\int_{z=-h/2}^{z=+h/2}\left(\sigma_x\delta\varepsilon_x + \tau_{zx}\delta\gamma_{zx}\right)dx\,dz - \int_{x=0}^{x=L} q(x)\,\delta w\,dx = 0 \qquad (5)$$

where the symbol δ denotes the variational operator. Employing Green's theorem in Eqn. (4) successively, we obtain the coupled Euler-Lagrange equations which are the governing differential equations and associated boundary conditions of the beam. The governing differential equations obtained are as follows:

$$EI\frac{d^4w}{dx^4} - \frac{24}{\pi^3}EI\frac{d^3\phi}{dx^3} = q(x) \qquad (6)$$

$$\frac{24}{\pi^3} EI \frac{d^3 w}{dx^3} - \frac{6}{\pi^2} EI \frac{d^2 \phi}{dx^2} + \frac{GA}{2} \phi = 0 \quad (7)$$

The associated consistent natural boundary condition obtained is of following form:
At the ends $x = 0$ and $x = L$

$$V_x = EI \frac{d^3 w}{dx^3} - \frac{24}{\pi^3} EI \frac{d^2 \phi}{dx^2} = 0 \text{ or } w \text{ is prescribed} \quad (8)$$

$$M_x = EI \frac{d^2 w}{dx^2} - \frac{24}{\pi^3} EI \frac{d\phi}{dx} = 0 \quad \text{or} \quad \frac{dw}{dx} \text{ is prescribed} \quad (9)$$

$$M_a = EI \frac{24}{\pi^3} \frac{d^2 w}{dx^2} - \frac{6}{\pi^2} EI \frac{d\phi}{dx} = 0 \text{ or } \phi \text{ is prescribed} \quad (10)$$

C. The general solution of governing equilibrium equations of the Beam

The general solution for transverse displacement $w(x)$ and warping function $\phi(x)$ is obtained using Eqns. (6) and (7) using method of solution of linear differential equations with constant coefficients. Integrating and rearranging the first governing Eqn. (6), we obtain the following equation

$$\frac{d^3 w}{dx^3} = \frac{24}{\pi^3} \frac{d^2 \phi}{dx^2} + \frac{Q(x)}{EI} \quad (11)$$

where $Q(x)$ is the generalized shear force for beam and it is given by $Q(x) = \int_0^x q\, dx + C_1$.

Now the second governing Eqn. (7) is rearranged in the following form:

$$\frac{d^3 w}{dx^3} = \frac{\pi}{4} \frac{d^2 \phi}{dx^2} - \beta \phi \quad (12)$$

A single equation in terms of ϕ is now obtained using Eqns. (11) and (12) as:

$$\frac{d^2 \phi}{dx^2} - \lambda^2 \phi = \frac{Q(x)}{\alpha EI} \quad (13)$$

where constants α , β and λ in Eqns. (11) and (12) are as follows :

$$\alpha = \left(\frac{\pi}{4} - \frac{24}{\pi^3} \right), \quad \beta = \left(\frac{\pi^3}{48} \frac{GA}{EI} \right) \text{and } \lambda^2 = \frac{\beta}{\alpha}$$

The general solution of Eqn. (13) is as follows:

$$\phi(x) = C_2 \cosh \lambda x + C_3 \sinh \lambda x - \frac{Q(x)}{\beta EI} \quad (14)$$

The equation of transverse displacement $w(x)$ is obtained by substituting the expression of $\phi(x)$ in Eqn. (12) and then integrating it thrice with respect to x. The general solution for $w(x)$ is obtained as follows:

$$EI\, w(x) = \iiiint q\, dx\, dx\, dx\, dx + \frac{C_1 x^3}{6} + \left(\frac{\pi}{4} \lambda^2 - \beta \right) \frac{EI}{\lambda^3} \quad (15)$$
$$\left(C_2 \sinh \lambda x + C_3 \cosh \lambda x \right) + C_4 \frac{x^2}{2} + C_5 x + C_6$$

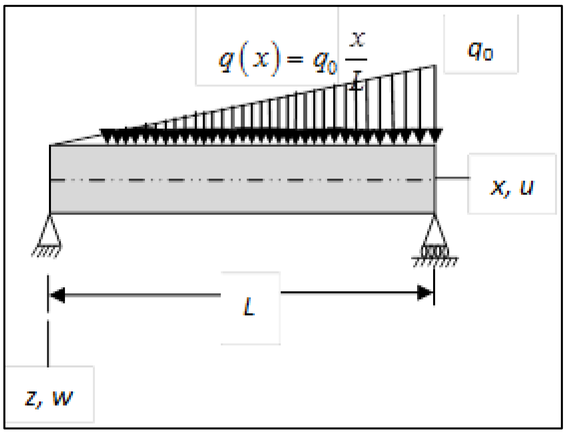
where C_1, C_2, C_3, C_4, C_5 and C_6 are arbitrary constants and can be obtained by imposing boundary conditions of beam.

IV. ILLUSTRATIVE EXAMPLES

In order to prove the efficacy of the present theory, the following numerical examples are considered. The material properties for beam used are: E = 210 GPa, μ = 0.3 and ρ = 7800 kg/m^3, where E is the Young's modulus, ρ is the density, and μ is the Poisson's ratio of beam material.

The beam has its origin at left hand side fixed support at x = 0 and free at x = L.

A. A simply supported beam with varying load, $q(x) = q_0 \frac{x}{L}$

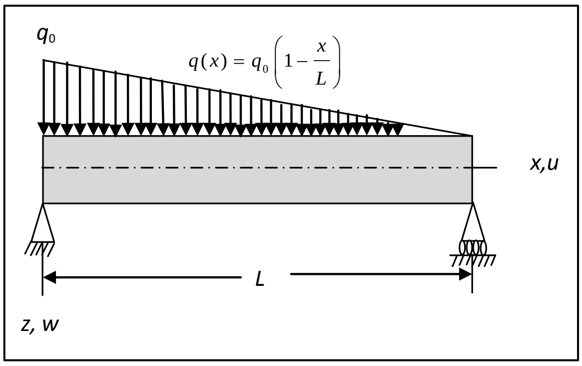


Expression for transverse shear stress using constitutive relationship τ_{zx}^{CR}

$$\tau_{zx}^{CR} = \left[\begin{matrix} \cos h \left(\frac{\alpha}{2} \right) - \\ \cos h \left(\frac{\alpha z}{h} \right) \end{matrix} \right] \left(\frac{A_0}{C_0} \frac{L}{h} \left(\begin{matrix} \frac{ch \lambda x}{\lambda L sh \lambda L} + \\ \frac{ch \lambda L}{2 \lambda L sh \lambda L} - \frac{x^2}{2L^2} - \frac{1}{6} \end{matrix} \right) \right)$$

B. A simply supported beam with varying load,

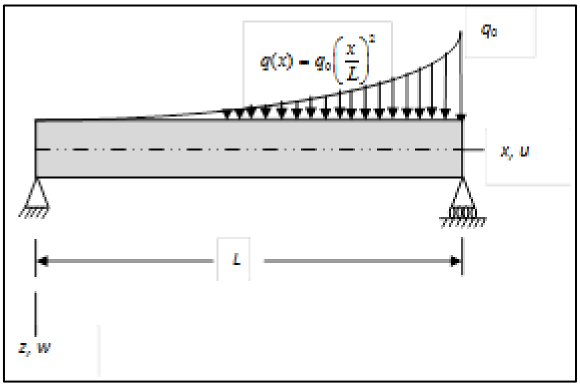
$$q(x) = q_0 \left(1 - \frac{x}{L} \right)$$



$$\tau_{zx}^{CR} = \frac{A_0}{C_0} \frac{L}{h} \left[ch \left(\frac{\alpha}{2} \right) - ch \left(\frac{\alpha z}{h} \right) \right] \left(\begin{matrix} 1 + \zeta(x) - 2 \frac{x}{L} + \\ \frac{x^2}{L^2} \end{matrix} \right)$$

C. A simply supported beam with varying load,

$$q(x) = q_0 \left(1 - \frac{x}{L} \right)$$



$$\tau_{zx}^{CR} = \frac{L}{h}\frac{A_0}{C_0}\left[ch\left(\frac{\alpha}{2}\right) - ch\left(\frac{\alpha z}{h}\right)\right]\left[\frac{ch\lambda x}{\lambda L sh\lambda L} + \frac{ch\lambda L}{2\lambda L sh\lambda L} - \frac{x^3}{3L^3} - \frac{1}{24}\right]$$

V. Results & Discussion

The results for maximum transverse displacement and maximum transverse shear stresses are presented in the following non dimensional form for the purpose of presenting the results in this paper,

$$\bar{u} = \frac{Ebu}{qh}, \quad \bar{w} = \frac{10Ebh^3 w}{qL^4}, \quad \bar{\sigma}_x = \frac{b\sigma_x}{q}, \quad \bar{\tau}_{zx} = \frac{b\tau_{zx}}{q}$$

Source	*Model*	$\bar{\tau}_{zx}^{CR}$	
		S=4	*S*=10
Present	NHPSDT	-0.9423	-2.413
Ghugal and Sharma [4]	HPSDT	-0.9627	-2.458
Timoshenko [11]	FSDT	-0.1436	-0.897
Bernoulli-Euler	ETB	-	-

Table 1: Maximum Transverse Shears Stresses $\bar{\tau}_{zx}^{CR}$ (X=0.01L, Z =0.0) (Example 1)

Source	*Model*	$\bar{\tau}_{zx}^{CR}$	
		S=4	*S*=10
Present	NHPSDT	1.9611	4.9027
Ghugal and Sharma [4]	HPSDT	1.9937	4.9843
Timoshenko [11]	FSDT	0.3454	0.8631
Bernoulli-Euler	ETB	-	-

Table 2: Maximum Transverse Shears Stresses $\bar{\tau}_{zx}^{CR}$ (X=0.01L, Z =0.0) (Example 2)

Source	*Model*	$\bar{\tau}_{zx}^{CR}$	
		S=4	*S*=10
Present	NHPSDT	-0.4903	-1.2257
Ghugal and Sharma [4]	HPSDT	-0.4984	-1.2461
Timoshenko [11]	FSDT	-0.3662	-0.5723
Bernoulli-Euler	ETB	-	-

Table 3: Maximum Transverse Shears Stresses $\bar{\tau}_{zx}^{CR}$ (X=0.01L, Z =0.0) (Example 3)

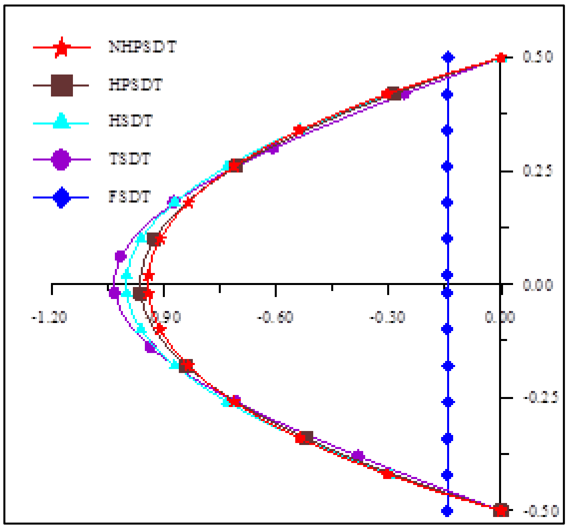


Fig. 2: Variation of axial displacement ($\bar{\tau}_{zx}^{CR}$) through the thickness of cantilever beam at (x =0.01 L, z) when subjected to cosine load for aspect ratio 4 (Example 1)

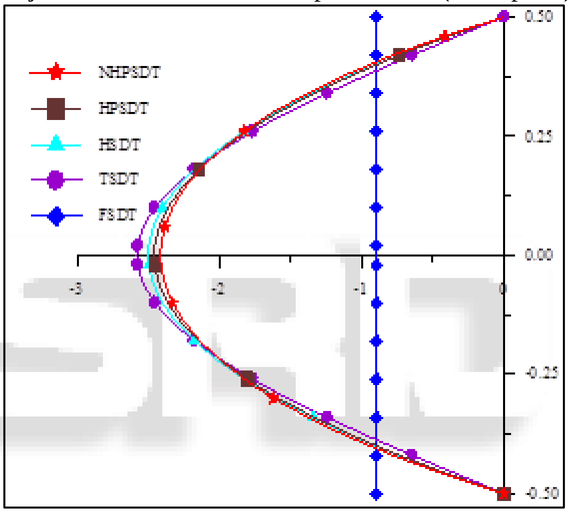


Fig. 3: Variation of axial displacement ($\bar{\tau}_{zx}^{CR}$) through the thickness of cantilever beam at (x =0.01 L, z) when subjected to cosine load for aspect ratio 10 (Example 1)

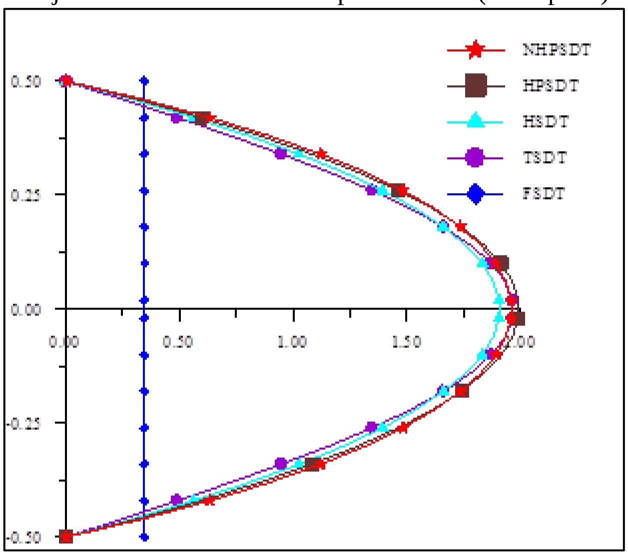


Fig. 4: Variation of axial displacement ($\bar{\tau}_{zx}^{CR}$) through the thickness of cantilever beam at (x =0.01 L, z) when subjected to cosine load for aspect ratio 4 (Example 2)

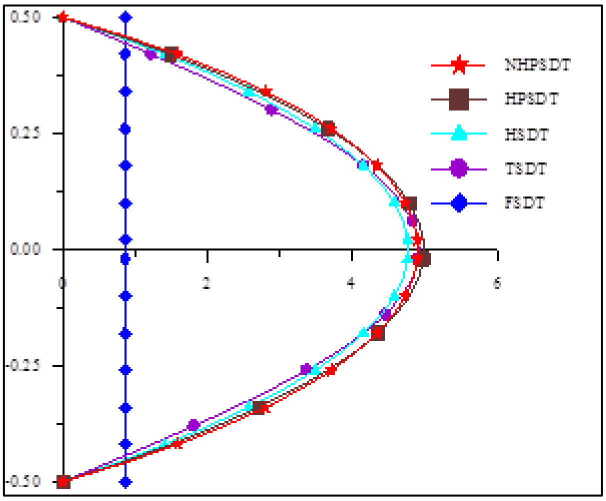


Fig. 5: Variation of axial displacement ($\bar{\tau}_{zx}^{CR}$) through the thickness of cantilever beam at (*x* =0.01 *L*, *z*) when subjected to cosine load for aspect ratio 10 (Example 2)

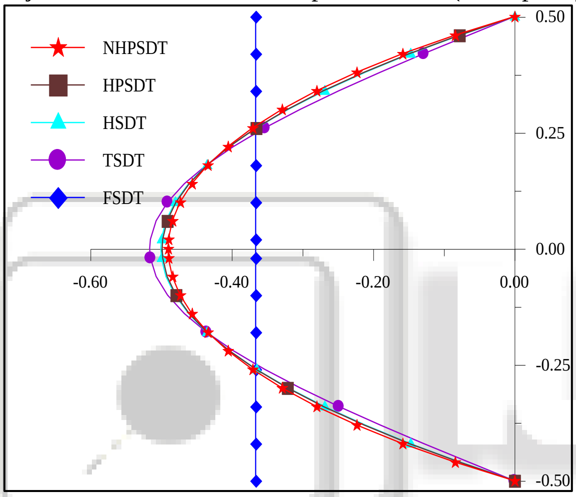


Fig. 6: Variation of axial displacement ($\bar{\tau}_{zx}^{CR}$) through the thickness of cantilever beam at (*x* =0.01 *L*, *z*) when subjected to cosine load for aspect ratio 4 (Example 3)

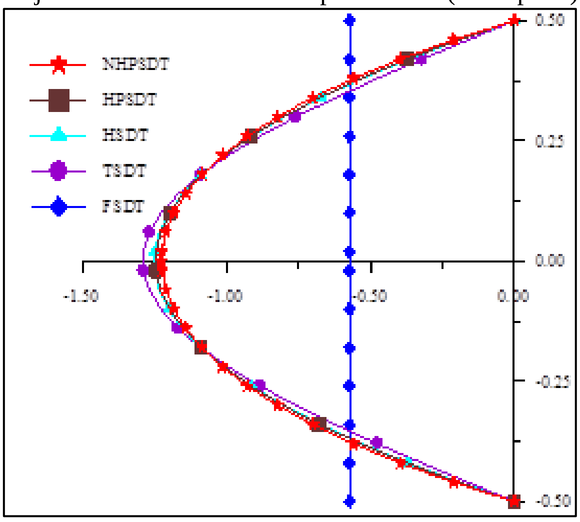


Fig. 7: Variation of axial displacement ($\bar{\tau}_{zx}^{CR}$) through the thickness of cantilever beam at (*x* =0.01 *L*, *z*) when subjected to cosine load for aspect ratio 10 (Example 3)

VI. DISCUSSION OF RESULTS

The transverse shear stresses ($\bar{\tau}_{zx}$) are obtained directly by constitutive relation and, alternatively, by integration of equilibrium equation of two dimensional elasticity and are denoted by ($\bar{\tau}_{zx}^{CR}$). The transverse shear stress satisfies the stress free boundary conditions on the top $(z = -h/2)$ and bottom $(z = +h/2)$ surfaces of the beam when these stresses are obtained by both the above mentioned approaches. The comparison of maximum non-dimensional transverse shear stress for a simply supported beam with varying load obtained by the present theory and other refined theories is presented in Tables I through III for aspect ratio of 4 and 10 respectively. The maximum transverse shear stress obtained by present theory using constitutive relation for aspect ratio 4 and for aspect ratio 10 results of present theory and HPSDT are in excellent agreement with each other. The

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