On Homogeneous Diophantine Equation $X^2 + Y^2 – XY = 7Z^2$

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Abstract— The ternary quadratic homogeneous equation representing homogeneous cone given by $X^2 + Y^2 – XY = 7Z^2$ is analyzed for its non-zero distinct integer points on it. Six different patterns of and special number patterns namely Polygonal number, Pyramidal number, Octahedral number and Nasty number are presented. Also knowing and integer solution satisfying the given cone, two triplex of integers generated from the given solution are exhibited.

Key words: Ternary Homogeneous Quadratic, Integral Solutions

I. Introduction

The ternary quadratic Diophantine equations offer an unlimited field for research due to their variety[1, 2]. For and extensive review of various problems, one may refer [3, 4]. This communication concerns with yet another interesting ternary quadratic equation X2 + Y2 – XY = 7Z2 representing a cone for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

A. Notations Used

– $T_{m,n}$ - Polygonal number of rank n with size m
– P_n^m - Pyramidal number of rank n with size m
– Pr_n - Pronic number of rank n
– OH_n- Octahedral number of rank n

II. Method of Analysis:

The ternary quadratic equation to be solved for its non-zero integer solution is

$X^2+Y^2–XY=7Z^2$ ----------(1)

The Substitution of the linear transformations

$X=u+v, Y=u-v, (u\neq0 v\neq0)$ ----------(2)

In leads to

$u^2 + 3v^2 = 7Z^2$ ----------(3)

Assume $Z(a, b) = a^2 + 3b^2 = 7Z^2$ $(a, b \neq 0)$ ----------(4)

We illustrate below six different patterns of non-zero distinct integer solutions to (1).

A. Pattern 1:

Write 7 as

$7 = (2 + i\sqrt{3})(2 - i\sqrt{3})$ ----------(5)

Substitute (4) and (5) in (3) and Employing the method of factorization, define

$$(u + i\sqrt{3}v) = (2 + i\sqrt{3})(a + i\sqrt{3}b)^2$$

Equating real and imaginary parts

$u = 2a^2 - 6b^2 - 6ab$
$v = a^2 - 3b^2 + 4ab$

Substituting the above values of u and v in (2), the non-zero distinct integer values for X and Y satisfying (1) are given by

$X = X(a, b) = 3a^2 - 9b^2 - 2ab$ ----------(6)

$Y=Y(a, b) = a^2 - 3b^2 - 10ab$ ----------(7)

Thus (4), (6) and (7) represent non-zero distinct integral solutions of (1) in two parameters.

1) Properties:

1) $X(A, A(A+1)) + 36T_{3,A}^2 - 3T_{4,A} = -4P_A^5$
2) $Y(A(A+1), (A+2)) - 4T_{3,A}^2 + T_{8,A} + 60P_A^3 \equiv 12 \pmod{14}$
3) $X(A, 1) - 3Y(A, 1) + Z(A,1) - T_{4,A} \equiv 3 \pmod{28}$
4) $X(A, 2) - 3T_{4,A} \equiv 9 \pmod 4$
5) $Y(1, B) + T_{8,B} \equiv 1 \pmod{12}$
6) $X(1, B) + T_{20,B} \equiv 3 \pmod{10}$
7) $Y(A, 1) - T_{4,A} \equiv 3 \pmod{10}$
8) $X(2B, 2) - T_{26,A} \equiv 0 \pmod 3$
9) $X(A+1, A+1) – T_{18,A} \equiv 8 \pmod{23}$
10) $-6X(2A, A) = 6*A^2$ a Nasty number

B. Pattern:2

Instead of (5) write 7 as

$7=(-2 + i\sqrt{3})(-2 - i\sqrt{3})$ ----------(8)

Following the procedure presented in pattern:1, the corresponding values of X and Y obtained from (2) are

$X = X(a, b) = -a^2 + 3b^2 -10ab$ ----------(9)

$Y = Y(a, b) = -3a^2 + 9b^2 - 2ab$ --------(10)

Thus (4), (9) and (10) represent non-zero distinct integral solutions of (1) in two parameters.

1) Properties:

1) $X(A, A(A + 1)) + T_{4,A} + 20P_A^5 = 12T_{3,A}^2$
2) $3X(A, 1) - Y(A, 1) + Z(A, 1) - T_{4,A} \equiv 6 \pmod 8$
3) $Y(A, 1) + T_{8,A} - 4A = 9$
4) $X(1, B) - T_{8,B} \equiv -2 \pmod 8$
5) $X(2, 2A) + T_{26,A} \equiv -4 \pmod{29}$
6) $Y(2, B) - T_{20,B} \equiv 0 \pmod 4$
7) $X(A, 2) \equiv 0 \pmod 2$
8) $X(A + 1, A + 1) + T_{18,A} \equiv 8 \pmod{23}$
9) $X(A(A + 1), (A + 2)) + T_{3,A}^2 - T_{8,A} + 60P_A^3 \equiv -4 \pmod 8$
10) $3X(A, A) = 24*A^2$ a Nasty number.

C. Pattern:3

Consider 7 as

$$7 = \frac{(5 + i\sqrt{3})(5 - i\sqrt{3})}{4}$$ --------(11)

For this choice, the corresponding values of X and Y obtained from(2) are represented

$X = X(a, b) = 3a^2 - 9b^2 + 7ab$ --------(12)

$Y = Y(a, b) = 2a^2 - 6b^2 - 8ab$ --------(13)

$Z=Z(a,b)=a^2+3b^2$ --------(14)

Which represent non-zero distinct integral solutions of (1) in two parameters.

1) Properties:

1) $Y(A, A(A + 1)) + 24T_{3,A}^2 + 16P_A^5 = T_{4,A}$
2) $X(A, 2) - T_{8,A} \equiv 0 \pmod{12}$
3) $Y(A(A + 1), (A + 2)) - 8T_{3,A}^2 + 48P_A^3 + T_{14,A} \equiv 24 \pmod{29}$
4) $X(A, 1) - T_{8,A} \equiv 0 \pmod 3$
5) $Y(1, B) + T_{14,B} \equiv 2 \pmod{13}$
6) $2X(A, 1) - 3Y(A, 1) + Z(A, 1) - T_{4,A} \equiv 3 \pmod{38}$
7) $Z(A, 1) - T_{4,A} \equiv 0 \pmod 3$
8) $6X(A, A) = 6*A^2$ a Nasty number
9) $3Z(A, A) = 12*A^2$ a Nasty number

D. Pattern:4

Let write 7 as

$$7 = \frac{(-5 + i\sqrt{3})(-5 - i\sqrt{3})}{4} \quad \text{--------(15)}$$

Following the analysis presented in pattern:3 and simplifying, the corresponding non-zero distinct integer solution of (1) are found to be

$X = X(a, b) = -4a^2 + 12b^2 - 16ab$ --------(16)
$Y = Y(a, b) = -6a^2 + 18b^2 + 4ab$ --------(17)
$Z=Z(a,b)=a^2+3b^2$ --------(18)

Thus (16), (17) and (18) represent non-zero distinct integral solutions of (1) in two parameters.

1) Properties:

1) $X(A, A(A + 1)) + 4T_{4,A} + 32P_A^5 = 48T_{3,A}^2$
2) $X(A(A + 1), (A + 2)) + 16T_{3,A}^2 - T_{26,A} + 96P_A^5 = 48 \pmod{59}$
3) $X(1, B) - T_{26,B} \equiv -4 \pmod 5$
4) $X(2, B) - T_{26,B} \equiv -16 \pmod{21}$
5) $Y(1, B) - T_{38,B} \equiv 6 \pmod{21}$
6) $Z(A, 1) - T_{4,A} \equiv 0 \pmod 3$
7) $X(A + 1, A + 1) + T_{18,A} \equiv 8 \pmod{23}$
8) $Y(2, B) - T_{38,B} \equiv -24 \pmod 5$
9) $3Y(A, A) = 12*4A^2$ a Nasty number

E. Pattern:5

The ternary quadratic equation (3) can be written as

$u^2 - 4Z^2) = 3(Z^2-v^2)$ --------(19)

Factorizing (19) we have

$(u+2Z)(u-2Z)=3(Z+v)(Z-v)$ --------(20)

Which is equivalent to the system of double equation

$(2B-3A)Z+Bu-3Av=0$ --------(21)
$(B+2A)Z–Au-Bv=0$ --------(22)

Applying the method of cross multiplication we get

$Z=-3A^2-B^2$ --------(23)
$u = -6A^2 + 2B^2 - 6AB$ -------(24)
$v = 3A^2 - B^2 - 4AB$ -------(25)

Substituting (23) and (24) in (2) we get

$X = X(A, B) = -3A^2 + B^2 - 10AB$ --------(26)
$Y = Y(A,B) = -9A^2 + 3B^2 - 2AB$ --------(27)

Thus (23), (26) and (27) represent non-zero distinct integral solutions of (1) in two parameters.

1) Properties:

1) $X(A, 1) + T_{8,A} \equiv 1 \pmod{12}$
2) $X(2, 2B) - T_{10,B} \equiv 12 \pmod{37}$
3) $Y(1, B) - T_{8,B} \equiv 0 \pmod 3$
4) $Y(B + 2, B + 2) - T_{18,B} \equiv 8 \pmod{23}$
5) $X(A, 2) + T_{8,A} \equiv 4 \pmod{22}$
6) $Y(2, B) -T_{20,B} \equiv 0 \pmod 4$
7) $X(A, 1) + Y(A, 1) + Z(A, 1) + T_{32,A} \equiv 3 \pmod{26}$
8) $-6Z(A, A) = 6*A^2$ a Nasty number.

F. Pattern:6

Also (20) is equivalent to the following two equation

$(-3A-2B)Z+Bu-3Av=0$ --------(28)
$(-2A+B)Z–Au–Bv=0$ --------(29)

Repeating the process as in pattern:5 the corresponding non-zero distinct integer solutions of (1) are given by

$u = 6A^2 - 12AB - 2B^2$ -------(30)
$v = 3A^2 + 4AB - 3B^2$ -------(31)

Substituting (30) and (31) in (2) we get

$X = X(A,B) = 9A^2 - 8AB - 5B^2$ --------(32)
$Y = Y(A,B) = 3A^2 - 16AB - B^2$ --------(33)

Thus (23), (32) and (33) represents non-zero distinct integral solutions of (1) in two parameters.

1) Properties:

1) $X(A,1) + T_{20,A} \equiv 0 \pmod 5$
2) $X(A, 2) - T_{20,A} \equiv 4 \pmod{16}$
3) $Y(1, B) +T_{4,B} \equiv 3 \pmod{16}$
4) $X(A + 1, A + 1) + T_{10,A} \equiv 4 \pmod{11}$
5) $Y(2A, 2) -T_{26,A} \equiv 4 \pmod{75}$
6) $Y(2, B) + T_{4,B} \equiv 0 \pmod 2$
7) $X(A, 1) + Y(A, 1) + Z(A, 1) - T_{20,A} \equiv 7 \pmod{16}$
8) $6X(A, A)=6*A^2$ a Nasty number.

III. REMARKABLE OBSERVATIONS

Employing the solution (X, Y, Z) of (1) each of the following expression among the special polygonal and pyramidal numbers are observed.

– $\frac{1}{7}\left\{\left(\frac{3P_{X-2}^3}{T_{3,X-2}}\right)^2 + \left(\frac{P_Y^5}{T_{3,Y}}\right)^2 - 2\left(\frac{3P_{X-2}^3}{T_{3,X-2}}\right)\left(\frac{P_Y^5}{T_{3,Y}}\right)\right\}$ is a perfect square.

– $\left(\frac{P_X^5}{T_{3,X}}\right)^2 + \left(\frac{3P_{Y-2}^3}{T_{3,Y-2}}\right)^2 - 3\left(\frac{P_X^5}{T_{3,X}}\right)^2\left(\frac{3P_{Y-2}^3}{T_{3,Y-2}}\right)^2 = 7\left(\frac{3P_{Z-1}^4}{T_{3,2(Z-1)}}\right)^2$

– $\left(\frac{P_X^5}{T_{3,X}}\right)^2 + \left(\frac{3P_{Y-2}^3}{T_{3,Y-2}}\right)^2 - 3\left(\frac{P_X^5}{T_{3,X}}\right)^2\left(\frac{3P_{Y-2}^3}{T_{3,Y-2}}\right)^2 \equiv 0 \pmod 7$

CONCLUSION

In this paper we have presented six different patterns of non-zero distinct integer solutions of the homogeneous cone given by $X^2 + Y^2 – XY = 7Z^2$. To conclude one may search for other patterns of solution and their corresponding properties.

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