Integral Solutions of the Homogeneous Quintic Diophantine Equation $x^5 - y^5 - x^2y^2 (x-y) = 972 (x-y) (z+w)^2p^2$

G. Janaki[1] C. Saranya[2]
[1,2]Assistant Professor
[1,2]Department of Mathematics
[1,2]Cauvery College for Women, Tiruchirappalli, Tamil Nadu India

Abstract— The Homogeneous Quintic Diophantine equation with five unknowns represented by $x^5 - y^5 - x^2y^2 (x-y) = 972 (x-y) (z+w)^2p^2$ is analyzed for its non-zero distinct integer solutions. Different patterns of integral solutions satisfying the equation are obtained. A few interesting relations between the solutions and some special numbers are presented.

Key words: Integral Solutions, the Homogeneous Quintic Diophantine Equation

NOTATIONS

- Obl_n = Oblong number of rank 'n'.
- P_n^m = Pyramidal number of rank 'n' with sides 'm'.
- $T_{m,n}$ = Polygonal number of rank 'n' with sides 'm'.
- CS_n = Centered Square number of rank 'n'.
- SO_n = Stella octangula number of rank 'n'.
- O_n = Octahedral number of rank 'n'.
- Gno_n = Gnomonic number of rank 'n'.
- $Star_n$ = Star number of rank 'n'.
- Tha_n = Thabit-ibn-kurrah number of rank 'n'.
- $Carl_n$ = Carol number of rank 'n'.
- Nex_n = Nexus number of rank 'n'.
- Mer_n = Mersenne number of rank 'n'.
- K_n = Kynea number of rank 'n'.
- J_n = Jacobsthal number of rank 'n'.
- j_n = Jacobsthal-Lucas number of rank 'n'.
- TO_n = Truncated Octahedral number of rank 'n'.
- TT_n = Truncated Tetrahedral number of rank 'n'.
- CH_n = Centered Hexagonal number of rank 'n'.
- $4DF_n$ = Four Dimensional Figurate number whose generating Polygon is a square.

I. INTRODUCTION

Mathematics is the language of patterns and relationships and is used to describe anything that can be quantified. Diophantine equations have stimulated the interest of various mathematicians. Diophantine equations with higher degree greater than three can be reduced in to equations of degree 2 or 3 and it can be easily solved. In [1-3], theory of numbers were discussed. In [4-5], quadratic diophantine equations are discussed. In [6-11], cubic, biquadratic and higher order equations are considered for its integral solutions.

In this communication a homogeneous quintic Diophantine equation, with five variables represented by $x^5 - y^5 - x^2y^2 (x-y) = 972 (x-y) (z+w)^2p^2$ is considered and in particular a few interesting relations among the solutions are presented.

II. METHOD OF ANALYSIS

The Quintic Diophantine equation to be solved for its non-zero integral solution is

$$x^5 - y^5 - x^2y^2 (x-y) = 972 (x-y) (z+w)^2p^2 \quad (1)$$

On substitution of the linear transformations,

$$\left.\begin{array}{l} x = u + v \ , \ y = u - v \\ z = u + 2v \ , \ w = u - 2v \end{array}\right\} \quad (2)$$

in (1) leads to,

$$u^2 + 3v^2 = 972\ p^2 \quad (3)$$

Four different patterns of non-zero distinct integer solutions to (1) are illustrated below:

A. Pattern 1

Assume $p = p(a,b) = a^2 + 3b^2 \quad (4)$

Where a and b are non-zero integers. and write

$$972 = (27 + 9i\sqrt{3})(27 - 9i\sqrt{3}) \quad (5)$$

Substituting (4) & (5) in (3), and using factorization method,

$$(u + i\sqrt{3}v)(u - i\sqrt{3}v) = (27 + 9i\sqrt{3})(27 - 9i\sqrt{3})\left((a + i\sqrt{3}b)^2 (a - i\sqrt{3}b)^2\right)$$

Equating the like terms and comparing real and imaginary parts, we get

$$u = u(a,b) = 27a^2 - 54ab - 81b^2$$
$$v = v(a,b) = 9a^2 + 54ab - 27b^2$$

Substituting the above values of u & v in equation (2), the corresponding integer solutions of (1) are given by

$$\left.\begin{array}{l} x = x(a,b) = 36a^2 - 108b^2 \\ y = y(a,b) = 18a^2 - 108ab - 54b^2 \\ z = z(a,b) = 45a^2 + 54ab - 135b^2 \\ w = w(a,b) = 9a^2 - 162ab - 27b^2 \\ p = p(a,b) = a^2 + 3b^2 \end{array}\right\} \quad (6)$$

1) Properties

$$y(a,a) - x(a,a) - p(a,a) + 5Star_a - CS_a + 16\ CH_a + 38\ Gno_a \equiv 0 \pmod{18}$$

$$w(a,a) - z(a,a) - a\ p(a,a) + SO_a + 3O_a + 12T_{12,a} + 7T_{26,a} \equiv 0 \pmod{125}$$

$$2y(2^n,1) - x(2^n,1) + 2\ Tha_n + 105\ j_n + 315\ J_n \equiv 0 \pmod{2}$$

$$z(2^n,1) + y(2^n,1) + p(2^n,1) - 30\ K_n - 25\ Carl_n + 118\ Mer_n \equiv 0 \pmod{222}$$

$$a^2 x(a,a) + y(a,a) + z(a,a) + 5\ Nex_a + 564(4DF_a) - 25\ SO_a + CH_a + 29\ T_{14,a} + 49\ Gno_a \equiv 0 \pmod{43}$$

B. Pattern 2

Instead of (5), write

$$972 = \frac{(81 + 117i\sqrt{3})(81 - 117i\sqrt{3})}{49} \quad (7)$$

Substituting (7) and (4) in (3) and employing the method of factorization, following the procedure presented in pattern 1, the corresponding integer solutions of (3) are represented by

$$u = u(a,b) = \frac{1}{7}\left(81a^2 - 702ab - 243b^2\right) \quad v = v(a,b) = \frac{1}{7}\left(117a^2 + 162ab - 351b^2\right)$$

Since our interest is on finding integer solutions, we have choose a and b suitably so that u and v are integers. Let us take a = 7A and b = 7B,

$$u = u(A,B) = 567A^2 - 4914AB - 1701B^2$$
$$v = v(A,B) = 819A^2 + 1134AB - 2457B^2$$

In view of (2), the integer solutions of (1) are given by

$$\left.\begin{aligned} x &= x(A,B) = 1386A^2 - 3780AB - 4158B^2 \\ y &= y(A,B) = -252A^2 - 6048AB + 756B^2 \\ z &= z(A,B) = 2205A^2 - 2646AB - 6615B^2 \\ w &= w(A,B) = -1071A^2 - 7182AB + 3213B^2 \\ p &= p(A,B) = 49A^2 + 147B^2 \end{aligned}\right\} \quad (7)$$

1) Properties

$$10\,TO_A + 18\,SO_A + 33(Star_A + 2CS_A) + 54\,Gno_A - A\,p(A,A) \equiv 0 \pmod{15}$$

$$z(A,A) - x(A,A) - A\,p(A,A) + 147\,O_A + 49\,SO_A + 42\,T_{26,A} \equiv 0 \pmod{462}$$

$$y(2^n,1) - p(2^n,1) + 43(4K_n + 3Carl_n) + 2981(Tha_n - Mer_n) \equiv 0 \pmod{308}$$

$$p(2^n,1) + 49(j_n + 3J_n - K_n) \equiv 0 \pmod{14^2}$$

$$x(A,A) + 5p(A,A) - y(A,A) + 2Star_A + 3CS_A + CH_A + T_{12,A} + 2Obl_A + 6Gno_A \equiv 0 \pmod{11}$$

C. Pattern 3

Instead of (5), write

$$972 = \frac{(135 + 99i\sqrt{3})(135 - 99i\sqrt{3})}{49} \quad (9)$$

Substituting (4) and (9) in (3), and employing the method of factorization, following the procedure presented in pattern 2, the corresponding integer solutions of (1) are represented by

$$\left.\begin{aligned} x &= x(A,B) = 1638A^2 - 2268AB - 4914B^2 \\ y &= y(A,B) = 252A^2 - 6048AB - 756B^2 \\ z &= z(A,B) = 2331A^2 - 378AB - 6993B^2 \\ w &= w(A,B) = -441A^2 - 7938AB + 1323B^2 \\ p &= p(A,B) = 49A^2 + 147B^2 \end{aligned}\right\} \quad (10)$$

1) Properties

$$A[z(A,A) - x(A,A)] - 1008\,P_A^5 + 252\,Gno_A + 63(Star_A + CS_A) \equiv 0 \pmod{126}$$

$$A^2\,p(A,A) - 38\,Nex_A - 72(4DF_A) + 187\,CS_A + 95(3O_A + SO_A) + 282\,Gno_A \equiv 0 \pmod{133}$$

$$TO_A + 45(3O_A + SO_A) + 6TT_A + 5CS_A + 9Gno_A - A\,p(A,A) \equiv 0 \pmod{5}$$

$$x(2^n,1) - z(2^n,1) + 84(Tha_n + Mer_n + j_n + 3J_n) + 693K_n \equiv 0 \pmod{1218}$$

$$p(1,2^n) + 147(Mer_n - Tha_n - Carl_n) \equiv 0 \pmod{14^2}$$

D. Pattern: 4

Writing $972 = \frac{(405 + 9i\sqrt{3})(405 - 9i\sqrt{3})}{169}$ (11)

The corresponding integer solutions of (1) are represented by

$$\left.\begin{aligned} x &= x(A,B) = 5382A^2 + 9828AB - 16146B^2 \\ y &= y(A,B) = 5148A^2 - 11232AB - 15444B^2 \\ z &= z(A,B) = 5499A^2 + 20358AB - 16497B^2 \\ w &= w(A,B) = 5031A^2 - 21762AB - 15093B^2 \\ p &= p(A,B) = 169A^2 + 507B^2 \end{aligned}\right\} \quad (12)$$

1) Properties

$$TO_A + 6TT_A - A\,p(A,A) + 5CS_A + 165(3O_A + SO_A) + 9Gno_A \equiv 0 \pmod{5}$$

$$13(Star_A + 2CH_A + Obl_A) - p(A,1) - 468\,Gno_A \equiv 0 \pmod{1079}$$

$$169(Carl_n + j_n + 3J_n) - p(2^n,1) \equiv 0 \pmod{26^2}$$

$$507(K_n - Tha_n + Mer_n) - p(1,2^n) \equiv 0 \pmod{26^2}$$

$$z(A,A) + w(A,A) - y(A,A) + 676\,Gno_A + 104(Star_A + CH_A + CS_A - 2Obl_A) \equiv 0 \pmod{364}$$

Note:

In addition, one may write 972 as

$$972 = \begin{cases} \frac{(216 + 18i\sqrt{3})(216 - 18i\sqrt{3})}{49} \\ \frac{(216 + 198i\sqrt{3})(216 - 198i\sqrt{3})}{169} \\ \frac{(189 + 207i\sqrt{3})(189 - 207i\sqrt{3})}{169} \end{cases}$$

For these choices, one may obtain different patterns of solutions of (1).

III. Conclusion

In this paper, we have made an attempt to determine different patterns of non-zero distinct integer solutions to the homogeneous Quintic Diophantine equation with five unknowns. As the equations are rich in variety, one may search for other forms of quintic equations with many variables and obtain their corresponding properties.

References

[1] Carmichael, R.D., The theory of numbers and Diophantine Analysis, Dover Publications, New York, 1959.

[2] Dickson L.E, History of Theory of Numbers, Vol.11, Chelsea Publishing company, New York,1952.

[3] Mordell. L.J, Diophantine equations, Academic Press,London,1969 Telang, S.G., Number theory, Tata Mc Graw Hill publishing company, New Delhi, 1996

[4] Gopalan.M.A., Vidhyalakshmi.S and Umarani.J., "On ternary Quadratic Diophantine equation $6(x^2+y^2)-8xy=21z^2$ ", Sch.J. Eng. Tech. 2(2A); 108- 112,2014

[5] Janaki.G and Saranya.C., Observations on the Ternary Quadratic Diophantine Equation $6(x^2+y^2)-11xy+3x+3y+9=72z^2$, International Journal of Innovative Research in Science, Engineering and Technology, Vol-5, Issue-2, Pg.no: 2060-2065, Feb 2016.

[6] Janaki.G and Vidhya.S., On the integer solutions of thehomogeneous biquadratic diophantine equation $x^4-y^4=82(z^2-w^2)p^2$, International Journal of Engineering Science and Computing, Vol. 6, Issue 6, pp.7275-7278, June, 2016.

[7] Gopalan.M.A and Janaki.G, Integral solutions of $(x^2-y^2)(3x^2+3y^2-2xy=2(z^2-w^2)p^3$, Impact J.Sci.,Tech., 4(1), 97-102, 2010.

[8] Janaki.G and Saranya.P., On the ternary Cubic Diophantine equation $5(x^2+y^2)-6xy+4(x+y)+4=40z^3$, International Journal of Science and Research- online, Vol 5, Issue 3, Pg.No:227-229, March 2016.

[9] Janaki.G and Saranya.C., Integral Solutions of the non-homogeneous heptic equation with five unknowns $6(x^2+y^2)-11xy+3x+3y+9=72z^2$, International Journal of Engineering Science and Computing, Vol. 6, Issue 5, pp.5347-5349, May, 2016.

[10] Janaki.G and Saranya.C., Integral Solutions of the ternary cubic equation $3(x^2+y^2)-4xy+2(x+y+1)=972\ z^3$, International Research Journal of Engineering and Technology, Vol. 4, Issue 3, pp.665-669, March, 2017.

[11] Janaki.G and Saranya.C., Integral Solutions of the homogeneous biquadratic diophantine equation $3(x^4-y^4)-2xy(x^2-y^2)=972\ (z+w)p^3$, International Journal for Research in Applied Science and Engineering Technology, Vol. 5, Issue 8, pp.1123-1127, Aug 2017.