Comparison of Range of Applicability of Rayleigh-Ritz Method and Method of Weighted Residual for Calculating the Deflection of a Simply supported Beam with Central Loading Condition

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Abstract— A beam is a structural element that is primarily resists the transverse load or laterally applied loads to the beam's axis. Beams are traditionally descriptions of building or structural elements, but any structures such as automotive automobile frames, aircraft components, machine frames, and other mechanical or structural systems contains beam structures that are designed to carry transverse loads. In engineering, deflection is the result to which a structural element is displaced under load. There are many method s to find out the deflection of a beam such as virtual work method, direct integration, Castiglione's method. Macaulay's method or the direct stiffness method but Leonhard Euler and Daniel Bernoulli in 1750, first time developed a theory for finding the deflection of beam. Though recent studies argue that Leonardo da Vinci was the first to make a observation but he couldn't develop a theory due to lack of Hooke's law and calculus. There is also different approximate methods were developed to calculate the deflection such as Point collaboration method, Sub-domain collaboration method, Galerkin Method, method of least square etc. In this investigation an attempt to be made a comparison of range of applicability of Rayleigh-Ritz method and above mentioned method of Weighted Residual to calculate the deflection of a simply supported beam under central loading. From the investigation, it is concluded that Rayleigh –Ritz method gives the accurate deflection result of the subjected beam.

Key words: Beam, Deflection, Simply Supported, Central Loading Condition, Rayleigh-Ritz Method, Method of Weighted Residual

I. INTRODUCTION

A beam is a structural member used for bearing loads generally transverse load. Beam can be classified into many type based on geometry, equilibrium conditions and the type of support. Based on geometry, there are four type of beams such as straight beam, curved beam, tapered beam and based on the shape of the cross section(I , C , T cross section etc). Based on equilibrium condition, one is statically determinate beam and other one is statically indeterminate beam. Based on support condition beam also can be classified, such as simply supported, cantilever, over hanged, continuous and fixed beam. Here in this study a simply supported beam is taken with a rectangular cross section. Both of Rayleigh-Ritz and all those four Weighted Residual was applied to find the deflection and compared those results.

II. STATEMENT OF THE PROBLEM

Here a rectangular cross section based beam (fig-1) is taken for the purpose of analysis. The width depth and length of that beam such that B mm, D mm and L mm. Elastic modules and the moment of inertia of the beam material such that E N/mm^2 and I mm^4. That load was applied to the beam such as P N. Then the deflection of the beam under P N load will be δ mm. From the both of the method analysis result was verified using following data of a same configuration beam, which is given in table-1.

Cross Section	Rectangular
Width(B)	100 mm
Depth(D)	150 mm
E	2×10^5 N/mm^2
L	1000 mm
I	2.8125×10^7 mm^4
P	50 N

Table 1: Property of a beam.

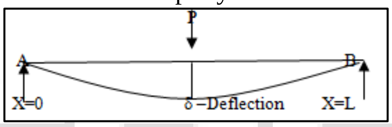

Fig. 1: Free Body Diagram of simply supported beam under concentrated loaded at the middle.

III. RAYLEIGH-RITZ METHOD

According to Rayleigh –Ritz method, when a body is loaded, its dimension is changed does some work is done on the body the and the induced work is absorbed and stored as strain energy in that body which may have some capacity for doing some other work. The amount of energy in a body due to the work done by the force applied on it within the elastic limit is called strain energy or potential energy. When that potential energy (Π) which is the sum of internal stain energy and external work is minimum then it's 1st derivative will be zero(ΔΠ=0).

A. Analysis of deflection of beam

From the Fig-1,

1) *Π= U-H*

Such that Y is the deflection of that simply supported beam subjected to centrally loaded.

So, Y=δ mm

Such that, $Y= a_1 \text{Sin}\left(\frac{\pi x}{L}\right) + a_2 \text{Sin}\left(\frac{3\pi x}{L}\right)$

For a Beam,

$$U = \int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx \qquad (1)$$

$$H= P\times Y$$

$$Y= a_1 \text{Sin}\left(\frac{\pi x}{L}\right) + a_2 \text{Sin}\left(\frac{3\pi x}{L}\right)$$

$$\text{Or, } \left(\frac{d^2y}{dx^2}\right)^2 = \frac{\pi^4}{4L^4}(a_1^{\,2} + 81\, a_2^{\,2})$$

Putting value of, $\left(\frac{d^2y}{dx^2}\right)^2$ in equation (1) we got,

$$U= \frac{EI}{4}\times\frac{\pi^4}{4L^3}(a_1^{\,2} + 81\, a_2^{\,2})$$

And H= P×Y

$$\text{Or, } H= P\times a_1 \operatorname{Sin}\left(\frac{\pi x}{L}\right) + a_2 \operatorname{Sin}\left(\frac{3\pi x}{L}\right)$$

$$\text{Now, } Y_{max} \;@\; X = \frac{L}{2}$$

$$\text{Or, } H= P[a_1 - a_2]$$

2) $\Pi = U - H$

$$\text{Or, } \Pi = \frac{EI}{4}\times\frac{\pi^4}{4L^3}(a_1^{\,2} + 81\, a_2^{\,2}) - P[a_1 - a_2]$$

According to Rayleigh –Ritz

$$\frac{d\pi}{da_1} = \frac{d\pi}{da_2} = 0 \qquad (2)$$

From the condition (2),

$$a_1 = \frac{2pL^3}{EI\pi^4} \quad \text{and } a_2 = -\frac{2pL^3}{81EI\pi^4}$$

$$Y= \frac{2pL^3}{EI\pi^4} \operatorname{Sin}\left(\frac{\pi x}{L}\right) - \frac{2pL^3}{81EI\pi^4} \operatorname{Sin}\left(\frac{3\pi x}{L}\right) \qquad (3)$$

Maximum deflection @ x= $\frac{L}{2}$ in eqn (3)

$$\delta\max = [a_1 - a_2]$$

$$\text{Or } \delta\max = \left[\frac{2pL^3}{EI\pi^4} + \frac{2pL^3}{81EI\pi^4}\right]$$

$$\text{Or } \delta\max = \frac{2pL^3}{EI\pi^4}\left(\frac{82}{81}\right)$$

$$\text{Or } \delta\max = .02\left(\frac{pL^3}{EI}\right) = \left(\frac{pL^3}{48EI}\right)$$

$$\delta 1 = \frac{pL^3}{48EI}$$

IV. Method of Weighted Residual

Method of Weighted Residual consist of four approximated method such as Point collaboration method, Sub-domain collaboration method, Galerkin Method, method of least square. All are this methods are different approaches to find out the deflection of the beam. Solution by each method is given below.

From the Euler and Bernoulli beam theory, moment equation of a structure can be represented by differential equation, is given below.

$$EI\frac{d^2y}{dx^2} + M = 0 \qquad (1)$$

Equation of (1) is the governing differential equation for the solution of the deflection of beam.

$$\text{Where } M= \frac{px}{2} - p\left(x - \frac{L}{2}\right)$$

Such that,

$$Y = a\sin\left(\frac{\Pi x}{L}\right)$$

$$\frac{d^2y}{dx^2} = - a\left(\frac{\pi}{L}\right)^2 \sin\left(\frac{\Pi x}{L}\right)$$

Putting the value of M and Y in the Eqn(1), we got

$$EI\frac{d^2y}{dx^2} + \frac{px}{2} - p\left(x - \frac{L}{2}\right) = 0$$

$$\text{Or,} EI\frac{d^2y}{dx^2} + \frac{px}{2} - p\left(x - \frac{L}{2}\right) = R(x) \text{ Or,} EI\left(- a\left(\frac{\pi}{L}\right)^2 \sin\left(\frac{\Pi x}{L}\right)\right) + \frac{px}{2} - p\left(x - \frac{L}{2}\right) = R(x)$$

A. Point Collaboration Method

In this method inside effects are set to zero for 'n' specific points. R(X) = 0

1) Solution

According to Point Collaboration Method

$$EI\left(- a\left(\frac{\pi}{L}\right)^2 \sin\left(\frac{\Pi x}{L}\right)\right) + \frac{px}{2} - p\left(x - \frac{L}{2}\right) = 0$$

$$\text{Or, } a = \frac{\frac{px}{2} - p\left(x - \frac{L}{2}\right)}{\left(\frac{\pi}{L}\right)^2 \sin\left(\frac{\Pi x}{L}\right))EI}$$

@x = L/2

$$a = \frac{pL^3}{39.48EI} = \delta 2$$

B. Sub-domain Collaboration Method

The domain is subdivided into 'n' sub-domains and the integral of the residual over each sub-domain is required to be zero. $\int_0^{L/2} R(x)\, dx = 0$

1) Solution

According to Sub-domain Collaboration method,

$$\int_0^{L/2} EI\left(-a\left(\frac{\pi}{L}\right)^2 \sin\left(\frac{\Pi x}{L}\right)\right) + \frac{px}{2} - p\left(x - \frac{L}{2}\right) dx = 0$$

$$\text{Or, } \frac{aEI\Pi}{L} = \frac{pL^2}{16}$$

$$\text{Or, } a = \frac{pL^3}{50.26EI} = \delta 3 \;@\; X= L/2$$

C. Galerkin Method

Here the domain integral of product of trial function with residual is set to zero.

$$\int_0^{L/2} YR(x)\, dx = 0$$

1) Solution

According to Galerkn Method,

$$\int_0^{L/2} YR(x)\, dx = 0$$

Putting the value of Y and $R(x)$ we got,

$$\int_0^{L/2} \left(a\sin\left(\frac{\Pi x}{L}\right)\right)\left(EI\left(- a\left(\frac{\pi}{L}\right)^2 \sin\left(\frac{\Pi x}{L}\right)\right) + \frac{px}{2} - p\left(x - \frac{L}{2}\right)\right) dx = 0$$

$$\text{Or, } EI(- a \times a)\left(\frac{\pi}{L}\right)^2 \frac{L}{2} = \frac{paL^2}{2\Pi}$$

$$\text{Or, } a = \frac{pL^3}{31.06EI} = \delta 4$$

D. Method of Least Square

The integral of the weighted square of residual over the domain is required to be minimum.

$$\int_0^{L/2} R^2(x)\, dx= I \text{ and } @ \frac{dI}{da} = 0 \text{ deflection } @ X= L/2$$

1) Solution

According to method of least square

$$\int_0^{L/2} \left(-EIa\left(\frac{\pi}{L}\right)^2 \sin\left(\frac{\Pi x}{L}\right) + \frac{px}{2} - p\left(x - \frac{L}{2}\right)\right)^2 dx = I$$

Such that,

$$A= -EIa\left(\frac{\pi}{L}\right)^2 \quad \text{and } B = \frac{p}{2}$$

$$\int_0^{L/2} \left(A\sin\left(\frac{\Pi x}{L}\right) - Bx + BL\right)^2 dx= I$$

$$\text{Or, } \left[\frac{A^2}{2}\left(\frac{L}{2} - \frac{L}{2\pi}\sin(\Pi)\right) + 2AB\left(-\frac{L^2}{2\Pi}\cos\left(\frac{\pi}{L}\right) + \left(\frac{L}{\Pi}\right)^2 \sin\left(\frac{\pi}{2}\right)\right) - B\frac{1}{3}(L/2)^3 + B(L^2)/2\right] = I$$

$$\text{Or, } a = \frac{pL^3}{62EI} = \delta 5$$

V. Comparison of Result

From the analytical investigation, comparative results are shown in the figure-2. From the figure 2 it is clearly shown that Rayleigh –Ritz method gave the accurate result of deflection compare to the method of weighted residual. And in figure-3 the percentage of error is given, In that figure-3 a comparison made to the actual deflection and the analysis result from the Rayleigh –Ritz method and method of weighted residual.

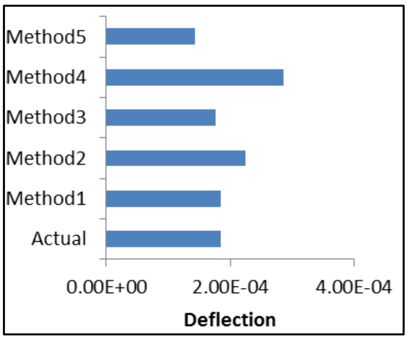

Fig. 2: Deflection equations from the analysis verified using the Table-1 property.

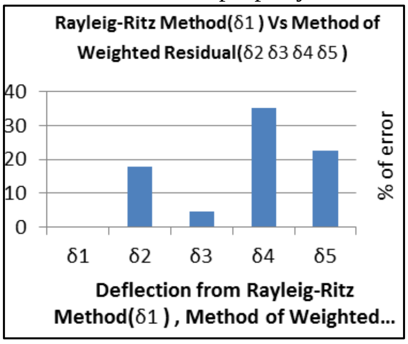

Fig. 3: Percentage of error compare to the actual deflection of subjected beam.

VI. Conclusion

In this paper the Analytical study is presented. From the analytical study on Rayleigh –Ritz method and method of weighted residual for finding the deflection of simply supported beam is given below.

– Rayleigh –Ritz method gives the accurate deflection result of the subjected beam.
– From figure-3 a comparison made to the actual deflection and the analysis result from the Rayleigh –Ritz method and method of weighted residual(Point collaboration method, Sub-domain collaboration method, Galerkin Method, method of least square) as follows 0%, 17.75%, 4.51616%, 35.29167%, 22.576 % error.

References

[1] R.H.Mohankar and et.al, "Comparative study of beam by flexibility method and slope deflection method", February 2016,IJSETR-volume 5-Issue-2
[2] Ashis Kumar and et.al, "Analysis of stress and deflection of cantilever beam and its validation using ANSYS", Jan-2016, AKSIJERA, Vol-6, issue-1, part-4
[3] S.Timosenko and D.H. Young, Strength of Material Book, Fifth Edition of East West PVT. LTD, India (Delhi-Madras), 1976.
[4] C.K. Wang, Intermediate Structural Analysis Book, 2010 edition of TaTa McGraw –Hill pvt ltd, Library of congress cataloging in publication data.
[5] Alaattin Aktas, "Determination of the deflection function of a composite cantilever beam using theory of anisotropic elasticity", 2001, Mathematical and computational application, vol-6, No-1,pp. 67- 74
[6] Biltu Mahato and et.al, "Experimental verification of deflection of beam using theoretical nad numerical approach", March 2015, IJAREST-volume 2, Issue-3.
[7] Mr.kumar Gaurav and et.al, "Prediction of deflection of cantilever beam of any arbitrary length using soft computation technique", March 2016,IJRASET , volume 4, Issue III