Fixed Point Theorem for Compatible Self-maps on Generalized Metric Space

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Abstract— The purpose of this paper is to establish a common fixed point theorem for a pair of compatible self-maps on D*-metric space and deduce corresponding results for metric spaces.

Key words: D*-Metric Spaces, Complete D*-Metric Space, Compatible. Associated Sequence

I. INTRODUCTION

Several researchers have made a significant contribution to the fixed point theorems of D-metric spaces in [1], [2], [3], [4] and [5].Recently in 1992 B. C. Dhage [6] has initiated a study of general metric spaces called D-metric spaces. As a probable modification of D-metric spaces, very recently, Shaban Sedghi, Nabi Shobe and Haiyun Zhou [7] have introduced. The notion of quasi-contractions on D*-metric spaces has been generalized by Brian Fisher [8]. Analogously we define generalized quasi-contractions among the self-maps of D*-metric spaces.

II. PRELIMINARIES

A. *Definition*

Let X be a non-empty set. A function $D^*: X^3 \to [0,\infty)$ is said to be a generalized metric or D*-metric on X, if it satisfies the following conditions:

- $D^*(x,y,z) \geq 0$ for all $x,y,z \in X$
- $D^*(x,y,z) = 0$ if and only if $x = y = z$
- $D^*(x,y,z) = D^*(\sigma(x,y,z))$ For all $x,y,z \in X$,
- Where $\sigma(x,y,z)$ is a permutation of the set $\{x,y,z\}$
- $D^*(x,y,z) \leq D^*(x,y,w) + D^*(w,z,z)$ for all $x,y,z,w \in X$.

The pair (X, D*), where D* is a generalized metric on X is called a D*-metric space or a generalized metric space.

B. *Definition*

A D*-metric space (X, D*) is said to be complete, if every Cauchy sequence in it converges in it.

C. *Definition*

Suppose f and g are selfmaps of a D*-metric space (X, D*). If $\lim_{n\to\infty} D^*(fgx_n, gfx_n, gfx_n) = 0$ for every sequence $\{x_n\}$ in X with $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$, then the pair (f, g) is said to be compatible.

1) *Remark*

Clearly, every commuting pair of selfmaps is a compatible pair, but not conversely as shown in the following example:

2) *Example*

Let $X = [0,4)$ and $D^*(x,y,z) = \max\{|x-y|,|y-z|,|z-x|\}$ for all $x,y,z \in X$. Then (X, D*) is a D*-metric space. Define $f: X \to X$ and $g: X \to X$ by $fx = \frac{x^2}{2}$ and $gx = \frac{x^2}{3}$ for all $x \in X$.

Now for any $x \in X$,

3) $gfx = g\,\frac{x^2}{2} = \frac{x^4}{12}$

and

4) $fgx = f\,\frac{x^2}{3} = \frac{x^4}{18}$

Thus *f* and *g* are not commuting.

Let $\{x_n\}$ be a sequence in X such that

5) $fx_n \to t$ *and* $gx_n \to t$ *as* $n \to \infty$,

So that $x_n^{\,2} \to 2t$ and $x_n^{\,2} \to 3t$ as $n \to \infty$, which is possible only if t = 0. Thus, whenever $fx_n \to t$ and $gx_n \to t$ as $n \to \infty$, we have t = 0 and also $x_n \to 0$ as $n \to \infty$.

Now for such sequence $D^*(fgx_n, gfx_n, gfx_n) = \left|\frac{x_n^{\,4}}{18} - \frac{x_n^{\,4}}{12}\right| = \left|\frac{x_n^{\,4}}{36}\right| \to 0$ as $n \to \infty$, proving f and *g* are compatible.

D. *Definition*

Let f and g be selfmaps of a non-empty set X and let $x_0 \in X$. If we can find a sequence $\{x_n\}$ in X satisfying that $fx_n = gx_{n-1}$ $n = 1, 2, 3, \ldots$ then the sequence $\{x_n\}$ will be called an associated sequence of x_0 relative to the self-maps f and g .

1) *Remark*

Observe that an associated sequence of X_0 relative the selfmaps f and g is not unique. Also note that, if f and g are selfmaps of X such that $g(X) \subseteq f(X)$, then every point $x_0 \in X$ has an associated sequence relative to the selfmaps f and g.

Further, if f and g are selfmaps of X, such that to each $x_0 \in X$, there is an associated sequence of x_0 relative to the selfmaps f and g, then it is easy to see that $g(X) \subseteq f(X)$. However it is possible to construct selfmaps f and g on X such that an associated sequence exists for some points of X only, as shown in example given below:

2) *Example*

Let $X = [0,1)$. Define $f: X \to X$ and $g: X \to X$ by $fx = \frac{1}{2}$ and gx = x for all $x \in X$. Then $f(X) = \left\{\frac{1}{2}\right\}$ and g(X) = X, showing that $g(X) \not\subset f(X)$. But for $x_0 = \frac{1}{2} \in X$, we get an associated sequence $\{x_n\}$ of x_0 relative to f and g, where $x_n = \frac{1}{2}$ for $n = 1,2,3,\ldots$.

Also for $x_0 \neq \frac{1}{2}$ in X, we cannot construct an associated sequence.

III. MAIN RESULT

A. Theorem

Suppose f is a continuous selfmap of a D*-metric space (X, D*). Then f has a fixed point in X if and only if there is a $\alpha \in (0,1)$ and a selfmap g of X such that

1) f and g are compatible

2) $D^*(gx, gy, gy) \le \alpha . D^*(fx, fy, fy)$ *For all* $x, y \in X$ *and*

3) There is a point $x_0 \in X$ *and an associate sequence* $\{x_n\}$ *of* X_0 *relative to the selfmaps f and g such that the sequence* $\{fx_n\}$ *converges to some point t of X. Further gt is the unique common fixed point of f and g.*

a) Proof

To prove the necessary part, suppose that f has a fixed point, say, $a \in X$ .Then $fa = a$. Define $g : X \to X$ by $gx = a$ for all $x \in X$. Now for any $x \in X$, $(gf)x = gfx = a$ And $(fg)x = fgx = fa = a$ for any $x \in X$

So that fg=gf. Since the commutativity implies the compatibility, f and g are compatible mappings.

Now for $\alpha \in (0,1)$ and for any $x, y \in X$, $D^*(gx, gy, gy) = D^*(a, a, a) = 0 \le \alpha . D^*(fx, fy, fy)$

Further an associated sequence of $x_0 = a$ relative to the selfmaps f and g is given by $x_n = a$ for $n = 0, 1, 2, \ldots$ and since the sequence $\{fx_n\}$ is a constant sequence converging to a, which is a point in X. Thus the conditions (3.1.1) to (3.1.3) of the theorem hold.

Conversely, suppose that there is a $\alpha \in (0,1)$ and a selfmap g of X such that the conditions (3.1.1) to (3.1.3) hold. Form (3.1.3) we have an associated sequence $\{x_n\}$ of x_0 relative to the selfmaps f and g such that $fx_n = gx_{n-1}$ for $n = 1, 2, 3, \ldots$ and $fx_n \to t$ as $n \to \infty$ for some $t \in X$. Then since $gx_n = fx_{n+1}$, it follows that $gx_n \to t$ as $n \to \infty$.

Now we shall show that g is continuous on X. To see this, suppose that {y_n} is a sequence in X with $y_n \to y$ as $n \to \infty$, $y \in X$.

Since f is continuous, $fy_n \to fy$ as $n \to \infty$. This together with the inequality $D^*(gy_n, gy, gy) \le \alpha . D^*(fy_n, fy, fy)$,

We get $D^*(gy_n, gy, gy) \le \alpha . D^*(fy_n, fy, fy) \to 0$ as $n \to \infty$, which implies that $gy_n \to gy$ as $n \to \infty$, showing g is continuous.

Using the continuity of f and g, we get $gfx_n \to gt$, $fgx_n \to ft$ as $fx_n \to t$. Since $fx_n \to t$, $gx_n \to t$ as $fx_n \to t$ and f and g are compatible, we have $\lim_{n \to \infty} D^*(fgx_n, gfx_n, gfx_n) = 0$ this implies that

$$D^*\left(\lim_{n \to \infty} fgx_n, \lim_{n \to \infty} gfx_n, \lim_{n \to \infty} gfx_n\right) = 0,$$

it follows that $D^*(ft, gt, gt) = 0$, which implies $ft = gt$.

To show that fg t = gf t, take $z_n = t$ for $n = 1, 2, 3, \ldots$, so that $fz_n \to ft$ and $gz_n \to gt$ as $n \to \infty$. Since f t=gt and f and g are compatible, we get $\lim_{n \to \infty} D^*(fgz_n, gfz_n, gfz_n) = 0$ this implies that

$$D^*\left(\lim_{n \to \infty} fgz_n, \lim_{n \to \infty} gfz_n, \lim_{n \to \infty} gfz_n\right) = 0,$$

Using the continuity of f and g, we get $gfz_n \to gft$ and $fgz_n \to fgt$ as $n \to \infty$. It follows that $D^*(fgt, gft, gft) = 0$ which implies $fgt = gft$. Consequently,

4) $fft = fgt = gft = ggt$

Now (3.1.2) and (3.1.4) gives

$$D^*(gt, ggt.ggt) \le \alpha . D^*(ft, fgt, fgt) = \alpha . D^*(gt, ggt, ggt)$$

And since $\alpha \in (0,1)$, it follows that $D^*(gt, ggt, ggt) = 0$. Thus $gt = ggt$. Using this in (3.1.4), we get $ggt = gt = fgt$, showing that gt is a common fixed point of f and g.

To see that f and g have unique common fixed point, suppose that $u = fu = gu$ and $v = fv = gv$ for some $u, v \in X$. Then $D^*(gu, gv, gv) \le \alpha . D^*(fu, fv, fv)$ gives $D^*(u, v, v) \le \alpha . D^*(u, v, v)$ which implies $D^*(u, v, v) = 0$, since $\alpha \in (0,1)$. Thus u = v. Therefore f and g have unique common fixed point.

B. Corollary

Suppose f is a continuous selfmap of a D*-metric space (X, D*). Then f has a fixed point in X if and only if there is a $\alpha \in (0,1)$ and a selfmap g of X such that

1) fg = gf

2) $D^*(gx, gy, gy) \le \alpha . D^*(fx, fy, fy)$ *for all* $x, y \in X$ *and*

3) there is a point $x_0 \in X$ *and an associate sequence* $\{x_n\}$ *of* x_0 *relative to the selfmaps f and g such that the sequence* $\{fx_n\}$ *converges to some point t of X.*

Further gt is the unique common fixed point of f and g.

a) Proof

Follows from Theorem 3.1 as commutativity implies the compatibility.

C. Consequences of Theorem 3.1

1) Theorem

Suppose *f* is a continuous selfmap of a *D**-metric space (X, D*). Then f has a fixed point in X if and only if there is a $\alpha \in (0,1)$ and a continuous selfmap g of X such that

a) *f* and *g* are compatible

b) $D^*(gx, gy, gy) \le \alpha . M(x, y)$ for all $x, y \in X$

Where

$$M(x, y) = M_{f,g}(x, y) = \max\{D^*(fx, fy, fy), D^*(fx, gy, gy), D^*(fy, gx, gx)\}$$

and

c) there is a point $x_0 \in X$ and an associated sequence $\{x_n\}$ of x_0 relative to the selfmaps f and g such that the sequence $\{fx_n\}$ converges to some point t of X.

Further gt is the unique common fixed point of f and g.

2) Proof

To prove the necessary part, suppose that f has a fixed point, say, $a \in X$, then $fa = a$. Define $g : X \to X$ by $gx = a$ for all $x \in X$, g is a constant function and hence continuous. Now for any $x \in X$,

$$(gf)x = gfx = a \text{ and}$$
$$(fg)x = fgx = fa = a \text{ for all } x \in X,$$

So that fg = gf. Since commutativity implies the compatibility, f and g are compatible mappings.

Now $\alpha \in (0,1)$ and for any $x, y \in X$,

$$D*(gx, gy, gy) = D*(a, a, a) = 0 \le \alpha.M(x, y)$$

The associated sequence $\{x_n\}$ of $x_0 = a$ relative to the selfmaps f and g is given by $x_n = a$ for $n = 1, 2, 3, \ldots$ and since $\{fx_n\}$ is a constant sequence converging to a, which is a point of X, the conditions (3.3.1) to (3.3.3) are satisfied.

Conversely, suppose that there is a $\alpha \in (0,1)$ and a continuous selfmap g of X satisfying the conditions (3.3.1) to (3.3.3). From (3.3.3), there is an associated sequence $\{x_n\}$ of x_0 such that $fx_n = gx_{n-1}$ for $n = 1, 2, 3, \ldots$ and $fx_n \to \infty$ as $n \to \infty$, it follows that $gx_n = fx_{n+1} \to t$ as $n \to \infty$. From (3.3.1) and since $fx_n \to t$, $gx_n \to t$ as $n \to \infty$, we have

$$\lim_{n \to \infty} D*(fgx_n, gfx_n, gfx_n) = 0,$$

Using the continuity of D*, $D*\left(\lim_{n\to\infty} fgx_n, \lim_{n\to\infty} gfx_n, \lim_{n\to\infty} gfx_n\right) = 0$ and also using the continuity of f and g, we get

$$D*(ft, gt, gt) = 0, \text{ so that } ft = gt.$$

To show that $fgt = gft$, take $z_n = t$ for $n = 1, 2, 3, \ldots$, so that $fz_n \to ft$ and $gz_n \to gt$ as $n \to \infty$ and since $ft = gt$ and f and g are compatible, we get $\lim_{n \to \infty} D*(fgz_n, gfz_n, gfz_n) = 0$

which implies that $D*\left(\lim_{n\to\infty} fgz_n, \lim_{n\to\infty} gfz_n, \lim_{n\to\infty} gfz_n\right) = 0$,

using the continuity of f and g, we get $gfz_n \to gft$ and $fgz_n \to fgt$ as $n \to \infty$, it follows that

$$D*(fgt, gft, gft) = 0, \text{ so that } fgt = gft. \text{ Consequently}$$

a) $fft = fgt = gft = ggt$

Now (3.3.2) and (3.3.4) give $D*(gt, ggt, ggt) \le \alpha.M(t, gt)$,

Where

$$M(t, gt) = \max\{D*(ft, fgt, fgt), D*(ft, ggt, ggt), D*(fgt, gt, gt)\}$$
$$= \max\{D*(gt, ggt, ggt), D*(gt, ggt, ggt), D*(ggt, gt, gt)\} = D*(gt, ggt, ggt)$$

That is $D*(gt, ggt, ggt) \le \alpha D*(gt, ggt, ggt)$. This implies that $D*(gt, ggt, ggt) = 0$, since $\alpha \in (0,1)$, and hence $gt = ggt$. Using this in (3.3.4), we get $ggt = gt = fgt$, showing that gt is a common fixed point of f and g.

To see that f and g have unique common fixed point, suppose that $u = fu = gu$ and $v = fv = gv$ for some $u, v \in X$. From (3.3.2), we have

$$D*(u, v, v) = D*(gu, gv, gv) \le \alpha.M(u, v),$$

Where

$$M(u, v) = \max\{D*(fu, fv, fv), D*(fu, gv, gv), D*(fv, gu, gu)\}$$
$$= \max\{D*(u, v, v), D*(u, v, v), D*(v, u, u)\}$$
$$M(u, v) = \max\{D*(u, v, v), D*(u, v, v), D*(u, v, v)\} = D*(u, v, v)$$

This implies that $D*(u, v, v) = 0$, and since $\alpha \in (0,1)$, and hence u = v, showing that f and g have unique common fixed point.

D. Corollary

Suppose *f* is a continuous selfmap of a *D**-metric space (X, D*). Then f has a fixed point in X if and only if there is a $\alpha \in (0,1)$ and a continuous selfmap g of X such that

1) fg = gf

2) $D*(gx, gy, gy) \le \alpha.M(x, y)$ *for all* $x, y \in X$

Where

$$M(x, y) = M_{f,g}(x, y) = \max\{ D*(fx, fy, fy), D*(fx, gy, gy), D*(fy, gx, gx) \}$$

and

3) there is a point $x_0 \in X$ *and an associated sequence* $\{x_n\}$ *of* x_0 *relative to the selfmaps f and g such that the sequence* $\{fx_n\}$ *converges to some point t of X.*

Further *gt* is the unique common fixed point of *f* and *g*.

a) Proof

Follows from Theorem 3.3 as commutativity implies the compatibility.

E. Corollary

Suppose f and g are selfmap of a D*-metric space (X, D*). Let f is continuous and if there is a $\alpha \in (0,1)$ and a positive integer k such that

1) fg = gf

2) $D*(g^k x, g^k y, g^k y) \le \alpha.D*(fx, fy, fy)$ *for all* $x, y \in X$

and

3) there is a point $x_0 \in X$ *and an associated sequence* $\{x_n\}$ *of* X_0 *relative to the selfmaps f and* g^k *such that the sequence* $\{fx_n\}$ *converges to some point t of X, then gt is the unique common fixed point of f and g.*

a) Proof

From (3.5.1), we get $fg^k = g^k f$. Thus f and g^k are commuting and hence satisfying the hypothesis of Corollary 3.4, and therefore f and g^k have unique common fixed point, say, b, then $g^k b = b = fb$, which implies that

$$g^k gb = g^{k+1} b = gg^k b = gb \text{ And } fgb = gfb = gb$$

This shows that gb is a common fixed point of f and g^k. The uniqueness of b implies that gb = b. Since $fb = b$, b is a common fixed point of *f* and *g*.

To prove that f and g have unique common fixed point, suppose that u = fu = gu and v = vu = gv for some $u, v \in X$. Then $u = fu = g^k u$ and $v = fv = g^k v$; which intern show that u and v are common fixed points of f and g^k. The uniqueness of common fixed points of f and g^k implies u = v.

References

[1] B. Ahmad, M. Ashraf and B.E. Rhoades, "Fixed point theorems for expansive mappings in D-metric spaces", Indian J. Pure and Applied Mathematics Vol-30, No.10, pp.1513-1518 (2001).

[2] B. C. Dhage, "Common fixed point principle in D-metric spaces", Bulletin of Calcutta Math. Soc. Vol-91, No. 6, pp 475-480 (1999)

[3] B. C. Dhage, A. M. Pathan and B. E. Rhoades, "A general extension principle for fixed point theorems in

D-metric spaces", Int. J. Math. & Math. Sci., Vol.23, pp.441-448 (2000)
[4] B.E. Rhoades, "A fixed point theorem for generalized metric spaces", Int. J. Math. Sci. 1 (2005).
[5] B. Singh and R.K. Sharma, "Common fixed points via compatible maps in D-metric space", Radovi Matemeticki 11 (2002); pp 145-153.
[6] B. C. Dhage, "Generalized metric spaces and mappings with fixed point", Bulletin of Calcutta Math. Soc. Vol-84, No. 4, pp 329-336(1992)
[7] Shaban Sedghi, Nabi Shobe and Haiyun Zhou, "A common fixed point theorem in D*-metric spaces", Journal of fixed point theory and applications, Haindavi Publishing corporation,
[8] B. Fisher, "Quasi-contractions on metric spaces", Proc. Amer. Math. Soc. 75 (1979), pp 51-54