Generalized Quasilinearization for PBVP for Hybrid Caputo Fractional Differential Equations

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Abstract—In this paper we study the method of Generalized Quasilinearization for PBVP for Caputo fractional differential equations with fixed moments of impulse through the solutions of linear IVPs. In order to prove this result we use the weakened assumption of C^q-continuity in place of local Hölder continuity.

Keywords: Hybrid Caputo Fractional Differential Equations, Generalized Quasilinearization, Existence
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I. Introduction

At present, there has been a significant amount of work done in the theory of fractional differential equations and a number of researchers are concentrating in this promising area due to its significant potential in applications related to Fluid Flow, Dynamical Processes in Self-Similar and Porous Structures, Diffusive Transport, Electrical Networks, Probability and Statistics, Control Theory of Dynamical Systems, Viscoelasticity, Electrochemistry of Corrosion, Optics and Signal Processing and many more areas. The works of Kilbas et al[2], Podlubny[1], Lakshmikantham et al[3] and the references [4-9] bear testimony to the continued interest in this area.

A lot of scope for the development of the theory of hybrid systems or impulsive differential systems has come due to the invaluable contribution of Lakshmikantham et al[10]. This is due to the fact that many evolution processes are characterized by the fact that they experience a change of state in a very short duration of time. This abrupt change can be considered as short term perturbations whose duration is negligible, that is these perturbations act instantaneously in the form of impulses. Hence hybrid systems form a better model to represent physical phenomena.

Combining these two areas of interest, the study of hybrid fractional differential equation or fractional differential equation with impulses have been initiated. Some of the papers in these areas are [14-20]. The method of quasilinearization [11] is a flexible mechanism that gives a sequence of iterations that converge quadratically to a solution. This technique has been fruitfullly utilized to obtain unique solutions of fractional differential equation with impulses. But in this case the function on the right hand side satisfies holder continuity. This hypothesis had been weakened to continuity and existence results for the impulsive or hybrid fractional differential equation through monotone iterative technique and quasilinearization have been studied [16,17,18,19,20]. In this paper, we develop the method of generalized quasilinearization for PBVP for hybrid Caputo fractional differential equations with the weakened hypothesis of C^q -continuity.

II. Preliminaries

The basic results that are needed to prove our main result are presented in this section. We begin with the definition of C_p – continuity, *R-L* fractional derivative, Caputo fractional derivative and proceed to state a lemma with the weakened hypothesis of C^q continuity. This lemma is essential in proving the basic differential inequality results. All these results are from [14].

As observed above, the comparison theorems [3], in fractional differential equations set-up require Hölder continuity. Although this requirement is used to develop iterative techniques such as the monotone iterative technique and the method of quasilinearization, there is no feasible way to check whether the functions involved are Hölder continuous. To avoid this situation, it has been shown in [14] that comparison results can be proved under the weaker condition of C_p – continuity. Lemma 2.3.1 in [3] is essential in establishing the comparison theorems, a detailed proof of this result under the weaker hypothesis was given in [14]. The basic differential inequality theorem, required comparison theorems and the lemma which are proved in [14] all are stated below.

We begin with the definition of the class $C_p[[0,T],\mathrm{P}]$.

A. *Definition 2.1*

m is said to be C_p continuous if $m \in C_p[[0,T],\mathrm{P}]$ that is $m \in C[(0,T],\mathrm{P}]$ and $(t-0)^p m(t) \in C[[0,T],\mathrm{P}]$ with p+q=1

B. *Definition 2.2*

For $m \in C_p[[0,T],\mathrm{P}]$, the Riemann-Liouville derivative of m(t) is defined as

$$D^q m(t) = \frac{1}{\Gamma(p)} \frac{d}{dt} \int_0^t (t-s)^{p-1} m(s)\,ds. \qquad (2.1)$$

We next state a lemma that is vital for our main result.

C. *Lemma 2.3*

Let $m \in C_p[[0,T],\mathrm{P}]$, Suppose that for any $t_1 \in [0,T]$, we havem(t_1)= 0 andm(t)< 0 for $0 \le t < t_1$, then it follows that

$$D^q m(t_1) \ge 0. \qquad (2.2)$$

It had been shown in [15] that results proved for Riemann-Liouville fractional differential equation hold for Caputo fractional differential equation but the converse is not true. Hence we state the following required results using Riemann-Liouville fractional differential equation.

We next state the fundamental fractional differential inequality result in the set up of Riemann-Liouville fractional derivative, with a weaker hypothesis from [14].

D. Theorem 2.4

Let $v, w \in C_p[[0,T],\mathbb{R}]$, $f \in C[[0,T]\times\mathbb{R},\mathbb{R}]$ *and* $(i)\ D^q v(t) \le f(t, v(t))$ and $(ii) D^q w(t) \ge f(t, w(t)),\ 0 < t \le T$, with one of the inequalities (i) or (ii)being strict. Then $v^0 < w^0$, where $v^0 = v(t)(t-0)^{1-q}|_{t=0}$ and $w^0 = w(t)(t-0)^{1-q}|_{t=0}$ implies that

$$v(t) < w(t),\ 0 \le t \le T. \tag{2.3}$$

The next result deals with the inequality theorem for non strict inequalities.

E. Theorem 2.5

Let $v, w \in C_p[[0,T],\mathbb{R}]$, $f \in C[[0,T]\times\mathbb{R},\mathbb{R}]$ *and* $(i)\ D^q v(t) \le f(t, v(t))$ and $(ii) D^q w(t) \ge f(t, w(t)),\ 0 < t \le T$. Assume f satisfies the Lipschitz condition

$$f(t,x) - f(t,y) \le L(x-y), \quad x \ge y,\ L > 0. \tag{2.4}$$

Then, $v^0 < w^0$, where $v^0 = v(t)(t-0)^{1-q}|_{t=0}$ and $w^0 = w(t)(t-0)^{1-q}|_{t=0}$, implies $v(t) \le w(t),\ t \in [0,T]$.

We now define a C^q-continuous function.

F. Definition 2.6

u is said to be C^q *continuous that is* $u \in C^q[[0,T],\mathbb{R}]$ *iff the Caputo derivative of u denoted by* ${}^cD^q u$ *exists and satisfies*

$${}^cD^q u(t) = \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q} u'(s)\,ds. \tag{2.5}$$

We note that the Caputo and Riemann-Liouville derivatives are related as follows:

$${}^cD^q x(t) = D^q[x(t) - x(t_0)]. \tag{2.6}$$

We choose to work with the Caputo fractional derivative, since the initial conditions for fractional differential equations are of the same form as those of ordinary differential equations. Further, the Caputo fractional derivative of a constant is zero, which is useful in our work. Consider the IVP for the Caputo fractional differential equation given by

$${}^cD^q x = f(t,x), \quad x(0) = x_0, \tag{2.7}$$

for $0 < q < 1,\ f \in C^q[[0,T]\times\mathbb{R}^n,\mathbb{R}^n]$.

If $x \in C^q[[0,T],\mathbb{R}^n]$ satisfies (2.7) then it also satisfies the Volterra fractional integral

$$x(t) = x_0 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, x(s))\,ds, \tag{2.8}$$

for $0 \le t \le T$.

We now state the comparison theorem for the Caputo fractional differential equation using the same weaker hypothesis. As the proof is similar to that of Theorem 2.4.3 in [3], we omit it.

G. Theorem 2.7

Assume that $m \in C^q[[0,T],\mathbb{R}]$ *and* ${}^cD^q m(t) \le g(t, m(t)),\ 0 \le t \le T$,

Where $g \in C[[0,T]\times\mathbb{R},\mathbb{R}]$. Let $r(t)$ be the maximal solution of the IVP

$${}^cD^q u = g(t,u),\ u(0) = u_0, \tag{2.9}$$

existing on $[0,T]$ such that $m(0) \le u_0$. Then we have $m(t) \le r(t),\ 0 \le t \le T$.

III. Impulsive Fractional Differential Equations

In this section, we begin with the basic definitions given in [15], where in the existence and stability results for hybrid Caputo fractional differential equation with fixed moments of impulse are studied.

A. Definition 3.1

Let $0 \le t_1 < t_2 < \dots < t_k < \dots$ *and* $t_k \to \infty$ *as* $k \to \infty$. *Then we say that* $h \in PC_p[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ *if* $h: (t_{k-1}, t_k] \times \mathbb{R}^n \to \mathbb{R}^n$ *is* C_p *-continuous on* $(t_{k-1}, t_k] \times \mathbb{R}^n$ *and for any* $x \in \mathbb{R}^n$

$$\lim_{(t,y)\to(t_k^+,x)} h(t,y) = h(t_k^+, x)$$

exists for $k = 1,2,\dots,\ n-1$.

B. Definition 3.2

Let $0 \le t_1 < t_2 < \dots < t_k < \dots$ *and* $t_k \to \infty$ *as* $k \to \infty$. *Then we say that* $h \in PC^q[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ *if* $h: (t_{k-1}, t_k] \times \mathbb{R}^n \to \mathbb{R}^n$ *is* C^q *- continuous on* $(t_{k-1}, t_k] \times \mathbb{R}^n$ *and for any* $x \in \mathbb{R}^n$

$$\lim_{(t,y)\to(t_k^+,x)} h(t,y) = h(t_k^+, x)$$

exists for $k = 1,2,\dots,\ n-1$.

Consider the hybrid Caputo fractional differential system defined by

$$\begin{cases} {}^cD^q x = f(t,x),\ t \ne t_k, \\ x(t_k^+) = I_k(x(t_k)),\ k = 1,2,3,\dots,\ n-1,\ t = t_k, \\ x(0) = x(T), \end{cases} \tag{3.1}$$

where $f \in PC[I \times \mathbb{R}^n, \mathbb{R}^n],\ I_k : \mathbb{R}^n \to \mathbb{R}^n,\ t \in I = [0,T]$ $k = 1,2,\dots,\ n-1$.

C. Definition 3.3

By a solution of the system (3.1) we mean a PC^q continuous function $x \in PC^q[[0,T],\mathbb{R}^n]$ such that

$$x(t) = \begin{cases} x_0(t, t_0, x_0),\ 0 \le t \le t_1, \\ x_1(t, t_1, x_1^+),\ t_1 < t \le t_2, \\ . \\ . \\ . \\ x_k(t, t_k, x_k^+),\ t_k < t \le t_{k+1}, \\ . \\ . \\ x_{n-1}(t, t_{n-1}, x_{n-1}^+),\ t_{n-1} < t \le T, \end{cases} \tag{3.2}$$

Where $0 \le t_1 < t_2 < \dots < t_{n-1} \le t_n = T$ and $x_k(t, t_k, x_k^+)$ is the solution of the PBVP of the fractional differential equation

$$\begin{cases} {}^cD^q x = f(t,x), \\ x_k^+ = x(t_k^+) = I_k(x(t_k)). \end{cases}$$

We begin by proving the basic differential inequality results in this setup, and we use the Theorem 2.4, Theorem 2.5 and Theorem 2.7 in [14] and the relation (2.6) between R-L and Caputo fractional derivatives. We observe that the fore mentioned theorems yield the required result over each sub-interval $(t_k, t_{k+1}]$, provided the corresponding hypothesis is satisfied.

Now we state the basic differential inequality result in this set up from [15].

D. Theorem 3.4

Let u,w $\in PC^q[[0,T],\mathrm{P}]$ *with*

$$\begin{cases} {}^cD^q v(t) \le f(t, v(t)),\ t \ne t_k, \\ v(t_k^+) \le I_k(v(t_k)),\ k = 1,2,3,\ldots,\ n-1,\ t = t_k, \\ v(0) \le v(T), \end{cases}$$

and

$$\begin{cases} {}^cD^q w(t) \ge f(t, w(t)),\ t \ne t_k, \\ w(t_k^+) \ge I_k(w(t_k)),\ k = 1,2,3,\ldots,\ n-1,\ t = t_k, \\ w(0) \ge w(T), \end{cases}$$

Where $f \in PC[I \times \mathrm{P}^n, \mathrm{P}^n]$ and f satisfies the hypothesis $f(t,x) - f(t,y) \le L(x-y),\ x \ge y,\ L > 0$ and I_k is a monotonically nondecreasing function of x. Then $v_0 < w_0$ implies that $v(t) \le w(t),\ t \in [0,T]$.

IV. GENERALIZED QUASILINEARIZATION

In this section, we obtain a solution for the PBVP for hybrid Caputo fractional differential equation through the solutions of linear IVPs.

We consider the hybrid Caputo fractional differential equation given by

$$\begin{cases} {}^cD^q x = f(t,x) + g(t,x),\ t \ne t_k, \\ x(t_k^+) = I_k(x(t_k)),\ k = 1,2,3,\ldots,\ n-1,\ t = t_k, \\ x(0) = x(T), \end{cases} \quad (4.1)$$

Where $f, g \in PC^q[[0,T] \times \mathrm{P}, \mathrm{P}],\ I_k; \mathrm{P} \to \mathrm{P}$ for each $k = 1,2,3,\ldots,\ n-1$.

We begin with the definition of natural lower and upper solutions for (4.1).

A. Definition 4.1

$\alpha, \beta \in PC^q[[0,T],\mathrm{P}]$ are said to be lower and upper solutions of equation (4.1) if and only if they satisfy the following inequalities,

$$\begin{cases} {}^cD^q \alpha = f(t,\alpha) + g(t,\alpha),\ t \ne t_k, \\ \alpha(t_k^+) = I_k(\alpha(t_k)),\ k = 1,2,3,\ldots,\ n-1,\ t = t_k, \\ \alpha(0) = \alpha(T), \end{cases} \quad (4.2)$$

and

$$\begin{cases} {}^cD^q \beta = f(t,\beta) + g(t,\beta),\ t \ne t_k, \\ \beta(t_k^+) = I_k(\beta(t_k)),\ k = 1,2,3,\ldots,\ n-1,\ t = t_k, \\ \beta(0) = \beta(T), \end{cases} \quad (4.3)$$

respectively.

We first state a couple of Lemmas that are necessary in the proof of our main theorem. The proofs are parallel to the Lemmas in [18].

B. Lemma 4.2

The linear non-homogeneous hybrid Caputo fractional differential equation

$$\begin{cases} {}^cD^q x = M(x-y) + f(t,y) + g(t,y),\ t \ne t_k, \\ x(t_k^+) = I_k(x(t_k)),\ k = 1,2,3,\ldots,\ n-1,\ t = t_k, \\ x(0) = x_0, \end{cases}$$

has a unique solution on the interval $[0,T]$.

Proof.

We proceed to prove the theorem in each subinterval.

Let $t \in [0, t_1]$ and consider the Caputo fractional differential equation

$$\begin{cases} {}^cD^q x = M(x-y) + f(t,y) + g(t,y), \\ x(0) = x_0. \end{cases}$$

Then from [2], we have that

$$x(t,0,x_0) = x(t) = x_0 E_q(M(t-0)^q) + \int_{t_0}^{t} (t-s)^{q-1} E_{q,q}(M(t-s)^q)\ f(s, y(s))ds$$

, $t \in [0, t_1]$ is the unique solution.

Then we have

$$x(t_1,0,x_0) = x(t_1) = x_0 E_q(M(t_1-0)^q) + \int_{t_0}^{t_1} (t_1-s)^{q-1} E_{q,q}(M(t_1-s)^q)\ f(s, y(s))ds$$

and $x(t_1^+) = I_1(x(t_1)) = x_1^+$ (say).

Now we consider the interval $(t_1, t_2]$ and the Caputo fractional differential equation

$$\begin{cases} {}^cD^q x = M(x-y) + f(t,y) + g(t,y), \\ x(t_1^+) = x_1^+. \end{cases}$$

Thus as earlier, the unique solution is given by,

$$x(t, t_1, x_1^+) = x(t)$$

$$= x_1^+ E_q(M(t-t_1)^q) + \int_{t_1}^{t} (t-s)^{q-1} E_{q,q}(M(t-s)^q)\ f(s, y(s))ds,\ t \in (t_1, t_2].$$

Then

$$x(t_2, t_1, x_1^+) = x(t_2)$$

$$= x_1^+ E_q(M(t_2-t_1)^q) + \int_{t_1}^{t_2} (t_2-s)^{q-1} E_{q,q}(M(t_2-s)^q)\ f(s, y(s))ds,\ t \in (t_1, t_2].$$

and

$$x(t_2^+) = I_2(x(t_2)) = x_2^+.$$

Then proceeding as earlier, we obtain the unique solution for the linear non-homogeneous hybrid Caputo fractional differential equation as

$$x(t) = \begin{cases} x_0(t,0,x_0),\ 0 \le t \le t_1, \\ x_1(t, t_1, x_1^+),\ t_1 < t \le t_2, \\ \quad . \\ \quad . \\ \quad . \\ x_{n-1}(t, t_{n-1}, x_{n-1}^+),\ t_{n-1} < t \le T. \end{cases} \quad (4.4)$$

C. Lemma 4.3

Suppose that

(i) $\alpha_0(t)$ and $\beta_0(t)$ are lower and upper solutions of the hybrid Caputo fractional differentialequation (4.1)

(ii)Let $\alpha_1(t)$ and $\beta_1(t)$ be the unique solutions of the linear non-homogeneous hybrid Caputo fractional differential equations

$$\begin{cases} {}^cD^q \alpha_1 = f(t,\alpha_0) + f_x(t,\alpha_0)(\alpha_1 - \alpha_0) + g(t,\alpha_0) + g_x(t,\beta_0)(\alpha_1 - \alpha_0),\ t \ne t_k, \\ \alpha_1(t_k^+) = I_k(\alpha_0(t_k)),\ k = 1,2,3,\ldots,\ n-1,\ t = t_k, \\ \alpha_1(0) = \alpha_0(T), \end{cases} \quad (4.5)$$

and

$$\begin{cases} {}^cD^q \beta_1 = f(t,\beta_0) + f_x(t,\alpha_0)(\beta_1 - \beta_0) + g(t,\beta_0) + g_x(t,\beta_0)(\beta_1 - \beta_0),\ t \ne t_k, \\ \beta_1(t_k^+) = I_k(\beta_0(t_k)),\ k = 1,2,3,\ldots,\ n-1,\ t = t_k, \\ \beta_1(0) = \beta_0(T). \end{cases} \quad (4.6)$$

(iii) I_k is nondecreasing function in x for each k=1,2,3,..., n-1.

(iv) f_x, g_x are continuous and Lipschitz on $[0,T]$.

Then $\alpha_0(t) \le \alpha_1(t) \le \beta_1(t) \le \beta_0(t)$ on $[0,T]$.

Proof.

Let $\alpha_0(t), \beta_0(t)$ be a lower and upper solutions of equations (4.2) and (4.3)

and $\alpha_1(t), \beta_1(t)$ be the unique solutions of equations (4.5) and (4.6) respectively.

To prove $\alpha_0(t) \le \alpha_1(t),\ t \in [0,T]$

set $p(t) = \alpha_0(t) - \alpha_1(t)$ where $t \in [0,t_1]$

Then we have

$${}^cD^q p(t) = {}^cD^q\alpha_0(t) - {}^cD^q\alpha_1(t)$$
$$\le f(t,\alpha_0) + g(t,\alpha_0) - [f(t,\alpha_0) + f_x(t,\alpha_0)(\alpha_1-\alpha_0) + g(t,\alpha_0) + g_x(t,\beta_0)(\alpha_1-\alpha_0)]$$
$$\le |f_x(t,\alpha_0) + g_x(t,\beta_0)|(\alpha_0-\alpha_1)$$
$$\le Mp(t)$$

Then

$${}^cD^q p(t) \le Mp(t) \tag{4.7}$$

also $p(0) = \alpha_0(0) - \alpha_1(0) \le 0.$

Then from the solution of the linear non-homogeneous hybrid Caputo fractional differential equation, we get

$$p(t) \le p(0)E_q(M(t-0)^q),\ t \in [0,t_1],$$

Since

$$p(0) \le 0$$

which yields

$$p(t) \le 0,\ \ t \in [0,t_1].$$

Thus we have

$$\alpha_0(t) \le \alpha_1(t),\ t \in [0,t_1],$$

and therefore we get

$$\alpha_0(t_1) \le \alpha_1(t_1).$$

from the assumption (iii) we obtain that

$$\alpha_0(t_1^+) \le I_1(\alpha_0(t_1)) = \alpha_1(t_1^+).$$

For $t \in (t_1,t_2]$ and consider the Caputo fractional differential equation

$$\begin{cases} {}^cD^q\alpha_1 = f(t,\alpha_0) + f_x(t,\alpha_0)(\alpha_1-\alpha_0) + g(t,\alpha_0) + g_x(t,\beta_0)(\alpha_1-\alpha_0), \\ \alpha_1(t_1^+) = \alpha_1(t_1^+) \end{cases}$$

Again setting $p(t) = \alpha_0(t) - \alpha_1(t)$, $t \in (t_1,t_2]$

we get ${}^cD^q p(t) = {}^cD^q\alpha_0(t) - {}^cD^q\alpha_1(t),\ t \in (t_1,t_2]$

that is, ${}^cD^q p(t) \le Mp(t)$ and $p(t_1^+) \le 0.$

Working as earlier, we get that $p(t) \le 0,\ t \in (t_1,t_2].$

From which we can conclude that

$$\alpha_0(t) \le \alpha_1(t),\ t \in (t_1,t_2].$$

gives

$$\alpha_0(t_2) \le \alpha_1(t_2)$$

and

$$\alpha_0(t_2^+) \le I_2(\alpha_0(t_2)) = \alpha_1(t_2^+).$$

Proceeding in a similar fashion over each subinterval $(t_k,t_{k+1}]$, we can show that

$$\alpha_0(t) \le \alpha_1(t),\ t \in [0,T].$$

Next to prove $\beta_1(t) \le \beta_0(t),\ t \in [0,T]$

set $p(t) = \beta_1(t) - \beta_0(t)$ where $t \in [0,t_1]$

$${}^cD^q p(t) = {}^cD^q\beta_1(t) - {}^cD^q\beta_0(t)$$
$$\le [f(t,\beta_0) + f_x(t,\alpha_0)(\beta_1-\beta_0) + g(t,\beta_0)$$
$$+ g_x(t,\beta_0)(\beta_1-\beta_0)] - [f(t,\beta_0) + g(t,\beta_0)]$$
$$\le |f_x(t,\alpha_0) + g_x(t,\beta_0)|(\beta_1-\beta_0)$$
$$\le Mp(t)$$

Then

$${}^cD^q p(t) \le Mp(t) \tag{4.8}$$

also

$$p(0) = \beta_1(0) - \beta_0(0) \le 0$$

Then from the solution of the linear non-homogeneous hybrid Caputo fractional differential equation, we get

$$p(t) \le p(0)E_q(M(t-0)^q),\ t \in [0,t_1],$$

since

$$p(0) \le 0$$

which yields

$$p(t) \le 0,\ \ t \in [0,t_1].$$

Thus we have

$$\beta_1(t) \le \beta_0(t),\ t \in [0,t_1],$$

and therefore we get

$$\beta_1(t_1) \le \beta_0(t_1)$$

from the assumption (iii) we obtain that

$$\beta_1(t_1^+) = I_1(\beta_0(t_1)) \le \beta_0(t_1^+).$$

For $t \in (t_1,t_2]$ and consider the Caputo fractional differential equation

$$\begin{cases} {}^cD^q\beta_1 = f(t,\beta_0) + f_x(t,\alpha_0)(\beta_1-\beta_0) + g(t,\beta_0) + g_x(t,\beta_0)(\beta_1-\beta_0), \\ \beta_1(t_1^+) = \beta_1(t_1^+) \end{cases}$$

Again setting $p(t) = \beta_1(t) - \beta_0(t)$, $t \in (t_1,t_2]$

we get ${}^cD^q p(t) = {}^cD^q\beta_1(t) - {}^cD^q\beta_0(t),\ t \in (t_1,t_2]$

that is, ${}^cD^q p(t) \le Mp(t)$ and $p(t_1^+) \le 0.$

Working as earlier, we get that $p(t) \le 0,\ t \in (t_1,t_2].$

From which we can conclude that

$$\beta_1(t) \le \beta_0(t),\ t \in (t_1,t_2].$$

gives

$$\beta_1(t_2) \le \beta_0(t_2)$$

and

$$\beta_1(t_2^+) = I_2(\beta_0(t_2)) \le \beta_0(t_2^+).$$

Proceeding in a similar fashion over each subinterval $(t_k,t_{k+1}]$, we can show that

$$\beta_1(t) \le \beta_0(t),\ t \in [0,T].$$

Further to prove $\alpha_1(t) \le \beta_1(t),\ t \in [0,T]$ set $p(t) = \alpha_1(t) - \beta_1(t)$, where $t \in [0,t_1]$.

Then we have

$${}^cD^q p(t) = {}^cD^q\alpha_1(t) - {}^cD^q\beta_1(t)$$
$$= f(t,\alpha_0) + g(t,\alpha_0) + f_x(t,\alpha_0)(\alpha_1-\alpha_0) + g_x(t,\beta_0)(\alpha_1-\alpha_0)$$
$$-[f(t,\beta_0) + g(t,\beta_0) + f_x(t,\alpha_0)(\beta_1-\beta_0) + g_x(t,\beta_0)(\beta_1-\beta_0)]$$
$$\le f_x(t,\alpha_0)(\alpha_0-\beta_0) + g_x(t,\beta_0)(\alpha_0-\beta_0) + f_x(t,\alpha_0)(\alpha_1-\alpha_0-\beta_1+\beta_0)$$
$$+ g_x(t,\beta_0)(\alpha_1-\alpha_0-\beta_1+\beta_0)$$
$$\le f_x(t,\alpha_0)(\alpha_0-\beta_0+\alpha_1-\alpha_0-\beta_1+\beta_0)$$
$$+ g_x(t,\beta_0)(\alpha_0-\beta_0+\alpha_1-\alpha_0-\beta_1+\beta_0)$$
$$\le |f_x(t,\alpha_0) + g_x(t,\beta_0)|(\alpha_1-\beta_1)$$
$$\le Mp(t)$$

Then

$${}^cD^q p(t) \le Mp(t) \tag{4.9}$$

also

$$p(0) = \alpha_1(0) - \beta_1(0) \le 0$$

Then from the solution of the linear non-homogeneous hybrid Caputo fractional differential equation, we get

$$p(t) \le p(0) E_q(M(t-0)^q), t \in [0, t_1],$$

which yields

$$p(t) \le 0, \; t \in [0, t_1].$$

Thus we have

$$\alpha_1(t) \le \beta_1(t), t \in [0, t_1],$$

and therefore we get

$$\alpha_1(t_1) \le \beta_1(t_1).$$

from the assumption (iii) we obtain that

$$\alpha_1(t_1^+) = I_1(\alpha_0(t_1)) \le I_1(\beta_0(t_1)) = \beta_1(t_1^+)$$

For $t \in (t_1, t_2]$ and consider the Caputo fractional differential equation

$$\begin{cases} {}^cD^q\alpha_1 = f(t,\alpha_0) + f_x(t,\alpha_0)(\alpha_1 - \alpha_0) + g(t,\alpha_0) + g_x(t,\beta_0)(\alpha_1 - \alpha_0), \\ \alpha_1(t_1^+) = \alpha_1(t_1^+) \end{cases}$$

and

$$\begin{cases} {}^cD^q\beta_1 = f(t,\beta_0) + f_x(t,\alpha_0)(\beta_1 - \beta_0) + g(t,\beta_0) + g_x(t,\beta_0)(\beta_1 - \beta_0), \\ \beta_1(t_1^+) = \beta_1(t_1^+) \end{cases}$$

Again setting $p(t) = \alpha_1(t) - \beta_1(t)$, $t \in (t_1, t_2]$

we get ${}^cD^q p(t) = {}^cD^q\alpha_1(t) - {}^cD^q\beta_1(t)$, $t \in (t_1, t_2]$

that is, ${}^cD^q p(t) \le Mp(t)$ and $p(t_1^+) \le 0$.

Working as earlier, we get that $p(t) \le 0, t \in (t_1, t_2]$.

From which we can conclude that

$$\alpha_1(t) \le \beta_1(t), t \in (t_1, t_2].$$

gives

$$\alpha_1(t_2) \le \beta_1(t_2)$$

and

$$\alpha_1(t_2^+) \le I_2(\alpha_1(t_2)) \le I_2(\beta_1(t_2)) = \beta_1(t_2^+).$$

Proceeding in a similar fashion over each subinterval $(t_k, t_{k+1}]$, we can show that $\alpha_1(t) \le \beta_1(t), t \in [0, T]$ and $\alpha_0(t_1^+) \le \alpha_1(t_1^+) \le \beta_1(t_1^+) \le \beta_0(t_1^+)$ hold.

Thus we have $\alpha_0(t) \le \alpha_1(t) \le \beta_1(t) \le \beta_0(t), t \in [0, T]$

We now state the main result of our paper.

D. Theorem 4.4

Assume that

(i) $\alpha_0, \beta_0 \in PC^q[[0, T], \mathrm{P}]$ be respectively natural lower and upper solutions of the PBVP for the hybrid Caputo fractional differential equation (12) such that $\alpha_0(t) \le \beta_0(t)$, $t \in [0, T]$.

(ii) Let $f, g \in PC^q[[0, T] \times \mathrm{R}, \mathrm{R}]$ and $f_x(t,x)$, $g_x(t,x)$ exists, *f* and *g* are nondecreasing and nonincreasing functions apply for each $t \in [0, T]$ such that

$$f(t,x) \ge f(t,y) + f_x(t,y)(x-y) \text{ and } g(t,x) \le g(t,y) + g_x(t,y)(x-y) \; x \ge y$$

and

$$|f_x(t,x) - f_x(t,y)| \le L_1|x-y|,$$

$L_1 \in \mathrm{P}$ and $|g_x(t,x) - g_x(t,y)| \le L_2|x-y|$, $L_2 \in \mathrm{P}$

(iii) I_k is increasing and Lipschitz in *x* for each $k = 1,2,3 \ldots n-1$.

Then there exist monotone sequences $\{\alpha_n\}, \{\beta_n\}$ such that $\alpha_n \to \rho, \beta_n \to r$ as $n \to \infty$ uniformly and monotonically to the unique solution $\rho = r = x$ of PBVP (4.1) on $[0, T]$ and the convergence is quadratic.

Proof.

Consider the linear hybrid Caputo fractional differential equation given by,

$$\begin{cases} {}^cD^q\alpha_{k+1} = f(t,\alpha_k) + f_x(t,\alpha_k)(\alpha_{k+1} - \alpha_k) + g(t,\alpha_k) + g_x(t,\beta_k)(\alpha_{k+1} - \alpha_k), t \ne t_k, \\ \alpha_{k+1}(t_k^+) = I_k(\alpha_k(t_k)), \; k = 1,2,3,\ldots, \; n-1, t = t_k \\ \alpha_{k+1}(0) = \alpha_k(T). \end{cases} \tag{4.10}$$

and

$$\begin{cases} {}^cD^q\beta_{k+1} = f(t,\beta_k) + f_x(t,\alpha_k)(\beta_{k+1} - \beta_k) + g(t,\beta_k) + g_x(t,\beta_k)(\beta_{k+1} - \beta_k), t \ne t_k, \\ \beta_{k+1}(t_k^+) = I_k(\beta_k(t_k)), \; k = 1,2,3,\ldots, \; n-1, t = t_k \\ \beta_{k+1}(0) = \beta_k(T). \end{cases} \tag{4.11}$$

Then it follows from Lemma 4.2 that the linear hybrid fractional differential equations (4.10) and (4.11) have unique solutions α_{k+1} and β_{k+1} respectively, whenever α_k and β_k are known lower and upper solutions of the hybrid caputo fractional differential equation (4.1).

Further by setting $k = 0$ in the above system and apply Lemma 4.3, we obtain that $\alpha_0 \le \alpha_1 \le \beta_1 \le \beta_0$ on $[0, T]$

We now claim that

$$\alpha_0 \le \alpha_1 \le \ldots \le \alpha_k \le \alpha_{k+1} \le \beta_{k+1} \le \beta_k \ldots \le \beta_1 \le \beta_0 \text{ on } [0, T]. \tag{4.12}$$

Since the result is already proved for $n = 0$, we assume that the result holds for $n = k$ and prove it for $n = k + 1$, this means that

$$\alpha_{k-1} \le \alpha_k \le \beta_k \le \beta_{k-1}, \tag{4.13}$$

where α_k and β_k are solutions of the hybrid Caputo fractional differential equation

$$\begin{cases} {}^cD^q\alpha_k = f(t,\alpha_{k-1}) + f_x(t,\alpha_{k-1})(\alpha_k - \alpha_{k-1}) + g(t,\alpha_{k-1}) + g_x(t,\beta_{k-1})(\alpha_k - \alpha_{k-1}), t \ne t_k, \\ \alpha_k(t_k^+) = I_k(\alpha_{k-1}(t_k)), \; k = 1,2,3,\ldots, \; n-1, t = t_k \\ \alpha_k(0) = \alpha_{k-1}(T). \end{cases} \tag{4.14}$$

and

$$\begin{cases} {}^cD^q\beta_k = f(t,\beta_{k-1}) + f_x(t,\alpha_{k-1})(\beta_k - \beta_{k-1}) + g(t,\beta_{k-1}) + g_x(t,\beta_{k-1})(\beta_k - \beta_{k-1}), t \ne t_k, \\ \beta_k(t_k^+) = I_k(\beta_{k-1}(t_k)), \; k = 1,2,3,\ldots, \; n-1, t = t_k \\ \beta_k(0) = \beta_{k-1}(T). \end{cases} \tag{4.15}$$

Now, using the relations (4.12), (4.13) and the hypothesis (ii), we get,

$$\begin{aligned} {}^cD^q\alpha_k &= f(t,\alpha_{k-1}) + f_x(t,\alpha_{k-1})(\alpha_{k-1} - \alpha_k) + g(t,\alpha_{k-1}) \\ &\quad + g_x(t,\beta_{k-1})(\alpha_{k-1} - \alpha_k) \\ &\le f(t,\alpha_k) + g(t,\alpha_k) \end{aligned}$$

and

$$\alpha_k(t_k^+) \le I_k(\alpha_{k-1}(t_k)) \le I_k(\alpha_k(t_k))$$

since I_k is an increasing function for each k.

This yields that α_k is a lower solution of (4.1) and further by Lemma 4.2, we obtain that α_{k+1} is a unique solution of (4.14) on $[0, T]$ and hence an application of the Lemma 4.3 yields that $\alpha_k \le \alpha_{k+1}$ on $[0, T]$

Similarly, it can be shown that β_k is an upper solution of (4.1) and by Lemma 4.2, that β_{k+1} is a unique solution of (4.15) on $[0, T]$ and hence an application of the Lemma 4.3 gives that $\beta_{k+1} \le \beta_k$ on $[0, T]$.

Further working in the lines of the Lemma 4.3, we obtain that $\alpha_{k+1} \le \beta_{k+1}$ on $[0, T]$

Hence by the principle of mathematical induction, we deduce the relation (4.12) and our claim holds. Clearly the sequences are piecewise uniformly bounded by relation (4.12), this also yields that the sequences $\{{}^cD^q\alpha_n\}$ and $\{{}^cD^q\beta_n\}$ are also piecewise uniformly bounded. By Lemma 2.3.2 in [3] we obtain that the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are equicontinuous in each subinterval $(t_k, t_{k+1}]$ and therefore by using Ascoli-Arzela Theorem, we conclude that there is a subsequence that converges uniformly on each subinterval $(t_k, t_{k+1}]$. Now since I_k is a continuous function (as lipschitz implies continuity) for each *k*, this convergence also holds at end points. Thus we obtain a sequence of piecewise C^q - continuous functions $\{\alpha_n(t)\}$ that converge uniformly to $\rho(t)$ on each subinterval $(t_k, t_{k+1}]$ and further $I_k(\alpha_k(t_k^+))$ converges uniformly to $I_k(\rho(t_k^+))$. Similarly, the sequence of iterations $\{\beta_n(t)\}$ converge uniformly to $r(t)$ in each subinterval $(t_k, t_{k+1}]$ and further $I_k(\beta_k(t_k^+))$ converges uniformly to $I_k(r(t_k^+))$.

Consider the corresponding hybrid Volterra fractional integrals, we can show that ρ and r are solutions of the PBVP(4.1)

Since f_x, g_x exists and is bounded on $[0,T]$, we obtain that f and g are Lipschitz and hence the solution is unique.

Thus $\rho = x = r$ on $[0,T]$.

Next, our aim is to show that this convergence is quadratic.

Set

$$p_{n+1} = x - \alpha_{n+1}$$

Thus for $t \neq t_k$,

$$ {}^cD^q p_{n+1} = {}^cD^q x - {}^cD^q\alpha_{n+1}$$

$$= f(t,x) + g(t,x) - f(t,\alpha_n) - g(t,\alpha_n) - [f_x(t,\alpha_n) - g_x(t,\beta_n)]\ (\alpha_{n+1} - \alpha_n)$$

$$\leq f_x(t,\xi)p_n + g_x(t,\eta)p_n + [f_x(t,\alpha_n) + g_x(t,\beta_n)](p_{n+1} - p_n)$$

Where $\alpha_n \leq \xi \leq x$ and $x \leq \eta \leq \beta_n$

Now using the increasing nature of f_x and the decreasing nature of g_x, we get that for $t \neq t_k$,

$$ {}^cD^q p_{n+1} \leq f_x(t,x)p_n + g_x(t,x)p_n - f_x(t,\alpha_n)p_n - g_x(t,\beta_n)p_n + f_x(t,\alpha_n)p_{n+1} + g_x(t,\beta_n)p_{n+1}$$

$$\leq L|p_n|_0^2 + Mp_{n+1}$$

where $M = Max(M_1, M_2)$ with $|f_x(t,\alpha_n)| \leq M_1$, $|g_x(t,\beta_n)| \leq M_2$ and $L = Max\ |L_1, L_2|$

for $t = t_k$, since I_k is Lipschitz for each k,

$$p_{n+1}(t_k^+) = x(t_k^+) - \alpha_{n+1}(t_k^+) = I_k(x(t_k)) - I_k(\alpha_{n+1}(t_k))$$

$$\leq K\ p_{n+1}(t_k)$$

Therefore $p_{n+1}(t_k^+) \leq Kp_{n+1}(t_k), t = t_k$.

Since $p_{n+1}(0) = 0$, we arrive at the hybrid Caputo fractional differential equation. Thus we have the hybrid Caputo fractional differential equation

$$ {}^cD^q p_{n+1} = L|p_n|_0^2 + Mp_{n+1}, t \neq t_k,$$

$$p_{n+1}(t_k^+) = K_k p_{n+1}(t_k), k = 1,2,3,\ldots n-1,$$

$$p_{n+1}(0) = 0.$$

Now using the solution of the linear non homogeneous fractional differential equation on each subinterval, we getfor $t \in (t_k, t_{k+1}]$.

$$p_{n+1}(t) = K_k \ldots K_3K_2K_1\frac{L|p_n|_0^2}{\Gamma(q+1)}(t_1-t_0)^q E_{q,q}(M(t_1-t_0)^q)E_q(M(t_2-t_1)^q)$$

$$E_q(M(t_3-t_2)^q)\ldots E_q(M(t_{k+1}-t_k)^q) + K_k\ldots K_3K_2$$

$$\frac{L|p_n|_0^2}{\Gamma(q+1)}(t_2-t_1)^q E_{q,q}(M(t_2-t_1)^q)E_q(M(t_3-t_2)^q)\ldots E_q(M(t_{k+1}-t_k)^q)$$

$$+K_k\ldots K_3\frac{L|p_n|_0^2}{\Gamma(q+1)}(t_3-t_2)^q E_{q,q}(M(t_3-t_2)^q)\ldots E_q(M(t_{k+1}-t_k)^q) + \ldots$$

$$+K_k\frac{L|p_n|_0^2}{\Gamma(q+1)}(t_k-t_{k-1})^q E_{q,q}(M(t_k-t_{k-1})^q)E_q(M(t_{k+1}-t_k)^q)$$

$$+\frac{L|p_n|_0^2}{\Gamma(q+1)}(t_{k+1}-t_k)^q E_{q,q}(M(t_{k+1}-t_k)^q)$$

$$\leq \frac{L|p_n|_0^2}{\Gamma(q+1)}\sum_{j=1}^{k+1}(t_j-t_{j-1})^q E_{q,q}(M(t_j-t_{j-1})^q)\prod_{i=j}^{k}K_iE_q(M(t_{i+1}-t_i)^q)$$

$$\leq \frac{L|p_n|_0^2}{\Gamma(q+1)}\sum_{j=1}^{k+1}l^q E_{q,q}[M(l)^q]\prod_{i=j}^{k}K_iE_q(M(l)^q)\ (since\ t_j - t_{j-1} = l)$$

$$\leq \frac{L|p_n|_0^2}{\Gamma(q+1)}\sum_{j=1}^{k+1}l^qK_j, K_{j+1},\ldots K_k[E_q(M(l^q))]^{k-j}E_{q,q}(M(l)^q)$$

$$\leq \frac{L|p_n|_0^2}{\Gamma(q+1)}\sum_{j=1}^{k+1}l^q\tilde{K}[E_q(M(l^q))]^{k-j}E_{q,q}(M(l)^q)$$

$$\leq \frac{L\tilde{K}}{\Gamma(q+1)}E_{q,q}(Ml^q)\Omega|p_n|_0^2$$

where

$$\tilde{K} = K_1 \ldots K_N$$

and

$$\Omega = \sum_{j=1}^{N}l^q[E_q(Ml^q)]^{k-j}$$

Thus

$$|p_{n+1}(t)| \leq \frac{L\tilde{K}}{\Gamma(q+1)}\Omega\ E_{q,q}(Ml^q)|p_n|_0^2$$

This implies the quadratic convergence of the sequence $\{\alpha_n(t)\}$.

Similarly, we can prove the quadratic convergence of the sequence $\{\beta_n(t)\}$ to the solution $x(t)$ of PBVP (4.1).

E. Remark

It can be observed that if we set $I_k \equiv 0$ for all k, then PBVP(4.1) reduces to Caputo fractional differential equation and generalized quasilinearization for there equations has been studied in [13].

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