A Study on Theoretical Solution for the Reynolds Stress in Canonical Flow Geometries

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Abstract— A theoretical approach is developed for solving the Reynolds stress in turbulent flows and is validated for canonical flow geometries. In nature, the turbulence characteristic leads to form the Reynolds stress, which is more complicated in Fluid dynamics. A model has been introduced to solve the Reynolds stress in turbulence modeling. A closed form solution for the Reynolds stress is found in terms of root variables such as velocity, velocity gradient, turbulence kinetic energy and a viscous term.

Key words: Turbulent Flow, Laminar Flow, Reynolds Number, Reynolds Stress

I. INTRODUCTION

In physics, fluid dynamics is a subdiscipline of fluid mechanics that deal with fluid flow-the science of fluid (liquids and gases) in motion. It has several subdiscipline itself, including aerodynamics (the study of air and other gases in motion) and hydrodynamics (the study of liquids in motion). Fluid dynamics has a wide range of applications, including calculating forces and moments on aircraft, determining the mass flow rate of petroleum through pipelines, predicting weather patterns, understanding nebulae in interstellar space and modeling fission weapon detonation. Some of its principles are even used in traffic engineering, where traffic is treated as a continuous fluid, and crowd dynamics.

II. PRELIMINARIES

A. Definition: Turbulent Flow

A complicated series of events takes place which eventually leads to a radical change of the flow character. In the final state the flow behavior is random and chaotic i.e chaotic changes in pressure and flow velocity.

At higher Reynolds number flows are turbulent. At values of the Reynolds number above the Re_{crit}. The motion becomes intrinsically unsteady even with constant imposed boundary condition.

B. Definition: Laminar Flow

The flow is smooth and adjacent layers of fluid slide past each other in an orderly fashion. At low Reynolds number flows are laminar. At values of the Reynolds number below the Re_{crit}. If the applied boundary conditions do not change with time then flow is steady.

C. Definition: Reynolds Number

The Reynolds number is the ratio of inertial forces to viscous forces within a fluid which is subjected to relative internal movement due to different fluid velocities, in what is known as boundary layer in the case of a bounding surface such as the interior of a pipe. A similar effect is created by the introduction of the stream of higher velocity fluid, such as the hot gases from a flame in air. This relative movement generates fluid friction, which is a factor in developing turbulent flow. Counteracting this effect is the viscosity of the fluid, which as it increases, progressively inhibits turbulence, as more kinetic energy is absorbed by a more viscous fluid. The Reynolds number quantifies the relative importance of these two types of forces for given flow conditions, and is a guide to when turbulent flow will occur in a particular situation.

The Reynolds number is defined as

$$Re=\frac{\delta v L}{\mu}$$

Where,

- ρ is the density of the fluid (SI units: kg/m^3)
- v is a characteristic velocity of the fluid with respect to the object (m/s)
- L is a characteristic linear dimension (m)
- μ is the dynamic viscosity of the fluid (pa·s or N·s/m^2 or kg/m·s)

D. Definition: Reynolds Stress

In fluid dynamics, the Reynolds stress is the component of the total stress tensor in a fluid obtained from the averaging operation over the Navi-Strokes equations to account for turbulent fluctuations in fluid momentum.

For a homogeneous fluid and an incompressible flow, the flow velocities are split into a steady mean value U with a fiuctuating component $u'(t)$

$$\text{i.e, } u(t) = U + u'(t)$$

Is called Reynolds decomposition with $u(X, t)$ being the flow velocity vector having components $u(t)$ in the x_i coordinate direction (with x_i denoting the components of the coordinate vector X). The mean velocity U are determined by either time averaging or spatial averaging depending on the flow under study. Further $u'(t)$ denotes the fluctuating ((turbulence) part of the velocity.

The components τ'_{ij} of the Reynolds stress tensor are defined as :

$$\tau'_{ij} = \rho\overline{u'_i u'_j}$$

With ρ the fluid density, taken to be non-fluctuating for this homogeneous fluid.

III. MATHEMATICAL FORMULATION

The current theoretical framework leads to the "integral formula" explicitly giving the Reynolds stress in terms of the "root", calculate turbulence parameters. In Reynolds-averaged Navier-Stokes (RANS) equation, non-linear terms involving turbulent fluctuations arise since the absolute velocity is decomposed into the mean (U) and fluctuating (u') component: $u = U + u'$. The non-linear terms that develop during the averaging process in the RANS are called the Reynolds stress, which involves time-averaged components of products of fluctuating velocities, $u'_i u'_j$. Here, we omit the bar aboveu'^2,$u'v'$,etc…,for simplicity, and take the fluctuation parameters to be time-averaged. Figuring out how the Reynolds stress is related to the mean and other "root" turbulent parameters has been the topic of numerous studies, for quite some time. However, we notice that the decomposition is necessary only in the absolute coordinate frame, and if we move or displace the control volume at the mean speed o the flow (see figure 1) then the mean velocity drops out of the momentum equation. That is, RANS is greatly simplified in the relative coordinate frame, or for a control volume moving at the mean velocity of the fluid. Therefore, the x-momentum equation, for an incompressible boundary-layer flow, becomes:

$$\frac{\partial u'}{\partial t}+\frac{\partial u'^2}{\partial x}+\frac{\partial (u'v')}{\partial y}=-\frac{1}{p}\frac{dp'}{dx}+v\frac{\partial^{2'}}{\partial y^2} \quad \ldots\ldots(1)$$

In equation (1), t, x, and y are the time and coordinates, while u', v', and p' are the turbulent fluctuations with respect to the mean.

If the time mean of the fluctuating velocity does not vary appreciably in time, then we can write a "steady-state" momentum equation, and solve for the gradient of the Reynolds stress.

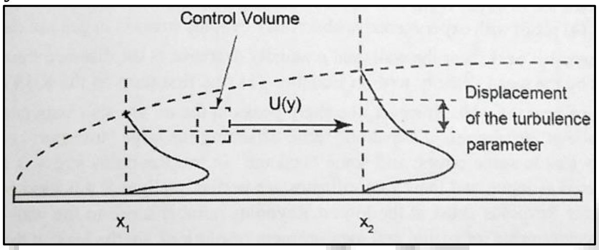

Fig. 1:

$$\frac{\partial (u'v')}{\partial y}=-\frac{\partial u'^2}{\partial x}-\frac{1}{p}\frac{dp'}{dx}+v\frac{\partial^{2'}}{\partial y^2} \quad \ldots\ldots(2)$$

In conventional calculations, the x-derivatives would have been set to zero for fully-developed flows, and we would be left with a triviality. However, we note the equations (1) and (2) have been written for a control volume which is moving along with a mean flow velocity, as shown in figure 1 for a boundary-layer flow as an example. For a flow over a flat plate, the boundary layer grows due to the "displacement" effect. The mass is displaced due to the fluid slowing down at the wall, as is the momentum and it turns out other turbulence parameters as well. The boundary layer thickness grows at a predictable rate, depending on the Reynolds number. Thus, if one rides with the fluid moving at the mean velocity, one would see a change in the all of the turbulence properties, as illustrated in figure 1. This displacement effect can be mathematically expressed as:

$$\frac{\partial}{\partial x}=C_1 U\frac{\partial}{\partial y} \quad \ldots\ldots(3)$$

The mean velocity, U, appears as a multiplicative factor in v equation(3). C_1 is a constant that depends on the Reynolds number. Similarly, the gradient of the pressure fluctuation will not be zero in general. However, this term is expected to be significant only for compressible flows, so we omit this term from further analysis in this phase of the work. In equation(2), we now have a simple integrable expression to find the Reynolds stress, after using equation(3). If we integrate by parts, we obtain:

$$u'v'=-C_1\left[Uu^2-c\int_0^y \frac{dU}{dy}u'^2 dy\right]+v_m\frac{\partial u'}{\partial y} \quad \ldots\ldots\ldots(4)$$

For axi-symmetric flow, the results are similar, leading to the following expression for the Reynolds stress.

$$u'v'=-C_1\left[Uu_z'^2-c\int_0^r \frac{dU}{dr}u_z'^2 dy\right]+v_m\left(\frac{\partial u_z'}{\partial r}+\int_0^r \frac{1}{r}\frac{\partial u_z'}{\partial r}dr\right) \quad \ldots\ldots\ldots(5)$$

IV. RESULTS & DISCUSSION

Figure 2 shows the comparison of the Reynolds stress obtained from equation(4) with experimental data. In that work, data on various turbulence quantities and the Reynolds stress (all normalized by the friction velocity) are provided, and also various scaling approaches tested with the data, in a well-designed experiment for flows over a flat plate with zero pressure gradient. The Reynolds number based on the momentum thickness (Re_θ) ranged from 1430 to 31,000. In equation(2), we use their measured parameters U and u'^2, and input them into equation(4). Gradients of U and u'^2, are calculated from the experimental data. As the experimental data are discontinuous, and at times hard to transcribe, there are some fluctuations and potential errors in the final calculations of the Reynolds stress, particularly close to the wall where the gradients are very steep and the data points all clustered. We can nonetheless input the root parameters into equation(4) to compute accordingly, and compare

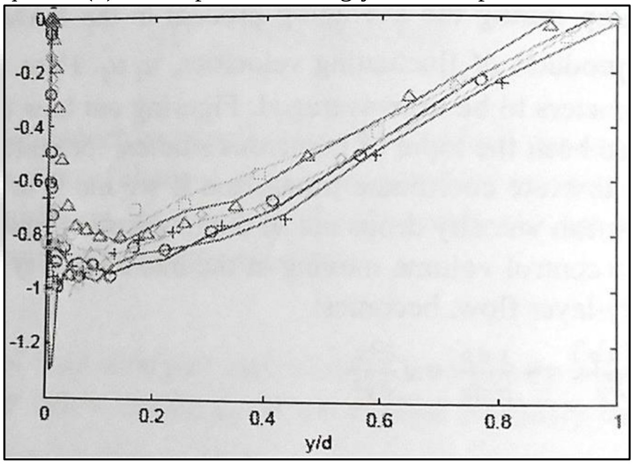

Fig. 2:

Figure 2 .Comparison of the Reynolds stress in boundary layer flows over a flat plate. The data (symbols) are for $(Re_\theta)=1430-31{,}000$. Lines represent current result (equation (4)), which are identified the color from pink (triangle data symbol) for $(Re_\theta)=1430$ to blue (circle data symbol) to $(Re_\theta)=31{,}000$.

With the measured Reynolds stress as in figure 2. In spite of dealing with discontinuous experimental data and their gradients, the comparison of equation (4) result with experimentally observed Reynolds stress is in general quite good.

u'^2 profiles have a sharp negative peak, near the wall, and gradually decreases as the distance from the wall increases. Although this is somewhat attenuated by the mean velocity term in Equation(1) (the first term on the RHS of Equation(4)), the resulting contribution of u'^2 is still high near the wall. However, the sharp peak in the u'^2 are also associated with a large gradient in u'^2 which increases the magnitude of the viscous "dissipation" term, offsetting the large "transport" effect. As noted above, dealing with experimental data that leads to some errors, and some "leakage" in the Reynolds stress is found in Figure 2. Unless the gradients are accurately entered as input, and their y coordinates are perfectly aligned, this kind of leakage can occur. Also ,the method is not able to track the Reynolds stress at the lowest Reynolds number close to the wall for the same reason, which is somewhat surprising at the requirements for spatial and measurement resolutions are the least at the condition.

Figure 3 is a similar comparison of Reynolds stress as calculated by Equation (4), with DNS results for fully-developed channel flows. DNS results are highly resolved and continuous, and therefore inputting the root parameters and taking the gradients do not lead to much fluctuations or misalignments, that was the case in Figure 2. We have performed DNS, using established methods for $Re_\tau = 110 - 650$, where Re_τ is the Reynolds number based on the friction velocity and channel half-width. The entire data set from the DNS is available on their website, including the mean velocity, turbulent fluctuating velocity components, and various moments of their products.

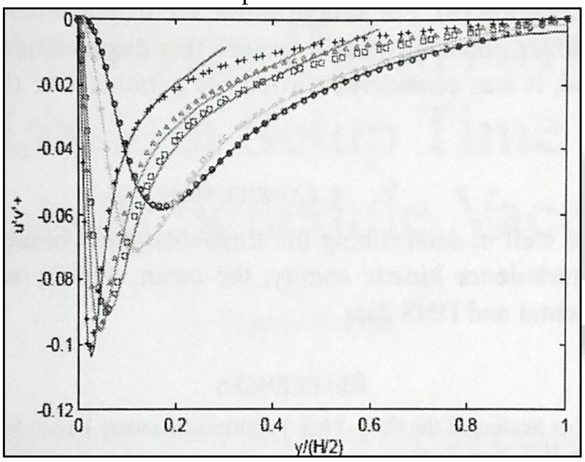

Fig. 3:

Figure 3, Comparison of the Reynolds stress in boundary layer flows over a flat plate. Lines are theoretical results, using Equation (4). Data symbols: circle ($Re_\tau = 110$), diamond (150), square (300), triangle (400),+(650). Line that track the data correspond to the same Reynolds number.

We input the necessary root turbulence parameters into Equation (4), and compare with the Reynolds stress from the DNS. The agreement is nearly perfect at low Reynolds number in Figure 3, which gives some confidence that we have captured the true physics of turbulent transport, and that the results are not a fortuitous coincidence. There is an interesting departure at higher Reynolds numbers, as the solutions starts to overshoot the DNS data as y approaches the centerline. The y location where this departure start to occur decreases (further away from the centerline) at higher Reynolds numbers. This departure is due to the fact that the symmetry boundary condition for channel flows, at the centerline, has not yet been imposed. As noted earlier, the integral term is a "displacement" term accumulating from the wall, and at centerline the displacement must cancel out. For example, the integral formula (Equation (4)) and its preceding transport (Equation (3)) have been derived for flows bounded on one side, such as the flow over a flat plate, and the solution proceeds from $y = 0$ (wall) onward. For channel flows, the flow is bounded on both sides, leading to the requisite symmetry condition at the centerline. One way to impose the symmetry boundary condition is to force the constant C_1 to be proportional to the velocity gradient.

For example,

$$C_1 = C_0 \left(\frac{\left(\frac{\partial U}{\partial y}\right)}{\left(\frac{\partial U}{\partial y}\right)_{y=0}} \right)^m \qquad \dots\dots\dots\dots (6)$$

With $m = 1/3$, indeed the calculated Reynolds stress tracks the DNS data fairly well at the Reynolds number of 400, as shown in Figure 4. There are small undulations now that derivatives of mean flow velocity are used in the multiplicative constant. Although this approach may not seem so elegant, Equation (3) and (4) have been derived based on the displacement of turbulence parameters, so that symmetry or other boundary conditions should be applied as in Equation (6). For low Reynolds numbers, the displacement apparently is insignificant and Equation (6) was not needed, as shown in Figure 3.

V. FINAL DISCUSSION

We have found a method to derive an expression for the Reynolds stress in simple flow geometries, leading to an "integral formula". In particular, if the data are continuous and aligned, then agreement is nearly perfect. There are some nuances and corrections that need to be examined, such as applying symmetry boundary conditions and relating the displacement effect to the flow geometry and Reynolds number.

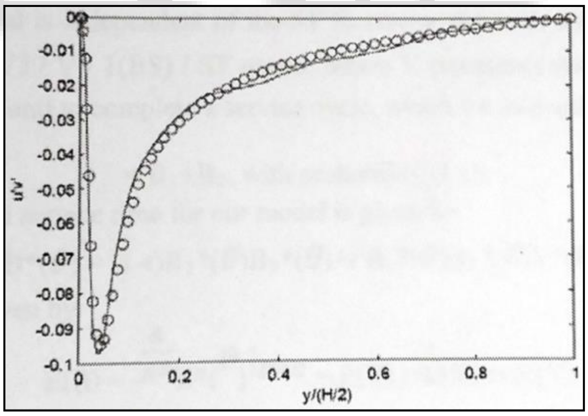

Fig. 4:

Thus far, the theory has been tested against relatively simple geometries. Would this method be useful in full three dimensional turbulent flows? That is a question that is being thought of at this time. The fact that $u'v'$ is related to u'^2 is easier to implement in computational applications as the turbulent kinetic energy can be related to u'^2, assuming isotropy, or an equation for u'^2 can be numerically solved in conjunction with Equation (4). For simple flows, the displacement effect could be effectively treated with Equation (6). Extensions to fully three- dimensional flows will require this displacement effect to be parameterized which may not be a simple matter. On the other hand, it was considered difficult to parameterize the Reynolds stress even in simple flows, for quite some time.

VI. Conclusion

In this, the method works quite well in determining the Reynolds stress based on input of root turbulence parameters, such as stream wise component of the turbulence kinetic energy, the mean velocity and its gradient. The predicted Reynolds stress is in good agreement with experimental and DNS data.

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