Two-Term Linear Fractional Nabla Difference Equations

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Abstract— In this present paper a study on factorial function, algebraic properties on Fractional Nabla Difference Equation. A new generalized Nabla difference equations is investigated from the Two-term linear Fractional Nabla Difference Equations. Some more special results and properties of Taylor's monomials are presented. Few relevant examples are also included and justify the proposed notions.

Key words: Gamma Function, Taylors Monomials, Linear Fractional, Nabla Difference Equations

I. INTRODUCTION

The fractional nabla difference calculus was initiated by Gray and Zhang, Anastassiou and Atici and Eloe**.** Here basic definitions, approaches and some properties of nabla fractional addition and differences were reported. In this article, we investigate the same for a class of linear fractional nabla difference equations involving Riemann-Liouville and caupto type fractional differences. We will illustrate the possible use of the N-transforms by applying it to solve some fractional order initial value problem. In particular, R-transform is the discrete transform defined on the integers for the delta (Δ) derivative and N-transform is the discrete transform defined on the integers for the nabla (∇) derivative. Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders.

II. PRELIMINARIES

A. *Definition 2.1*

For any $t \in \mathbb{R} \setminus \{\ldots,-2,-1,0\}$, the gamma function is defined by

$$\Gamma(t) = \int_0^\infty e^{-s} s^{t-1} ds, \text{For all } t > 0, \text{ and}$$

$$\Gamma(t+1) = t\Gamma(t).$$

B. *Definition 2.2*

For any $\alpha \in \mathbb{R}$ and $t \in \mathbb{R} \setminus \{\ldots,-2,-1,0\}$ such that $(t+\alpha) \in \mathbb{R} \setminus \{\ldots,-2,-1,0\}$, the rising factorial function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \qquad 0^{\overline{\alpha}} = 0.$$

C. *Definition 2.3*

Let $f : \mathbb{N}_a \to \mathbb{R}$, and $\alpha > 0$ be given. Then the α^{th}-order nabla fractional sum of f is given by

$$\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s)$$

For $t \in \mathbb{N}_a$. Where $\rho(s) = s-1$. Also, we define the trivial sun by $\nabla_a^{-0} f(t) = f(t)$ for $t \in \mathbb{N}_a$.

D. *Definition 2.4*

Let $f : \mathbb{N}_a \to \mathbb{R}$, and $\alpha > 0$ be given, and $N \in \mathbb{N}$ be chosen such that $N-1 < \alpha \le N$. Then the α^{th} – order Riemann-Liouville type nabla fractional difference of f is given by $\nabla_a^{\alpha} f(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{-\alpha-1}} f(s)$ for $t \in \mathbb{N}_{a+N}$.

and $\nabla_a^{\alpha} f(t) = \nabla^N \nabla_a^{-(N-\alpha)} f(t)$ for $t \in \mathbb{N}_{a+N}$.

For $\alpha = 0$, we set $\nabla_a^{0} f(t) = f(t)$ for $t \in \mathbb{N}_a$.

E. *Definition 2.5*

Let $f : \mathbb{N}_a \to \mathbb{R}$, and $\alpha > 0$ be given, and $N \in \mathbb{N}$ be chosen such that $N-1 < \alpha \le N$. Then the α^{th} – order Caputo type nabla fractional difference of f is given by

$$\nabla_{a*}^{\alpha} f(t) = \nabla_a^{\alpha} f(t) - \sum_{k=0}^{N-1} \frac{(t-a-k+1)^{\overline{k-\alpha}}}{\Gamma(k-\alpha+1)} \nabla_a^k f(a+k),$$

for $t \in \mathbb{N}_{a+N}$. and

$$\nabla_{a*}^{\alpha} f(t) = \nabla_a^{-(N-\alpha)}[\nabla^N f(t)]$$

For $t \in \mathbb{N}_{a+N}$.

For $\alpha = 0$, we set $\nabla_{a*}^{\alpha} f(t) = f(t)$,

For $t \in \mathbb{N}_a$.

III. TAYLOR MONOMIALS

The Taylor monomials are the functions $g_n : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $n \in \mathbb{N}_0$ and are defined recursively as follows: $g_0(t,s) \equiv 1$ for all $t,s \in \mathbb{T}$, and for $n \in \mathbb{N}_0$, $g_{n+1}(t,s) = \int_s^t g_n(\tau,s)\nabla_\tau$ for all $t,s \in \mathbb{T}$ .Now consider the discrete time scale $\mathbb{T} = \mathbb{N}_a = \{a, a+1, a+2, \ldots.\}$.

Where $a \in$ ¡ is fixed.
In this case, when $q(t) = q$,
where $q \in \{\Re / \Re = £\}$, and $t_0 = a$ then
we get

$$e_q(t,a) = (1-q)^{a-t},$$

and for all positive integers n (include 0),

$$g_n(t,a) = \frac{(t-a)^{\overline{n}}}{\Gamma n+1}$$

A. Definition 3.1

For any $t \in$ ¡ and $\alpha \in$ ¡ $\setminus\{\ldots,-2,-1\}$, the α^{th} - Taylor monomial is defined by,

$$g_\alpha(t,a) = \frac{(t-a)^{\overline{\alpha}}}{\Gamma(\alpha+1)}.$$

B. Result 3.2

For all $\alpha \in$ ¡ $\setminus\{\ldots.-2,-1,0,1,2,\ldots.\}$ and $n \in$ ¥$_0$, we have that

$$(1+\alpha)^{\overline{n}} = \frac{(-1)^n\Gamma(-\alpha)}{\Gamma(-(\alpha+n))}.$$

1) Properties of Taylor Monomials 3.3

1) $g_\alpha(t+1,t) = 1$,

for all $\alpha \in$ ¡ $\setminus\{\ldots.-2,-1,0,1,2,\ldots.\}$.

2) $g_\alpha(t,a) = g_\alpha(t-a,0)$,

for all $\alpha \in$ ¡ $\setminus\{\ldots.-2,-1,0,1,2,\ldots.\}$.

3) $g_\alpha(t+1,a) - g_{\alpha-1}(t+1,a) = g_\alpha(t,a)$,

for all $\alpha \in$ ¡ $\setminus\{\ldots.-2,-1,0,1,2,\ldots.\}$.

2) Proof

The function g : T $\to$ ¡, and the function is defined by $g_0(t,s) \equiv 1$ for all $t,s \in$ T.
Now, consider (1)

Take $$g_\alpha(t+1,t) = \frac{(t+1-t)^{\overline{\alpha}}}{\Gamma(\alpha+1)} = \frac{(1)^{\overline{\alpha}}}{\Gamma(\alpha+1)}$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(1)\Gamma(\alpha+1)} = \frac{1}{\Gamma(1)} = 1.$$

ie) $g_\alpha(t+1,t) = 1$,

For all $\alpha \in$ ¡ $\setminus\{\ldots.-2,-1,0,1,2,\ldots.\}$.

2) Take

$$g_\alpha(t,a) = \frac{(t-a)^{\overline{\alpha}}}{\Gamma(\alpha+1)} = \frac{((t-a)-0)^{\overline{\alpha}}}{\Gamma(\alpha+1)} = g_\alpha(t-a,0)$$

Hence $g_\alpha(t,a) = g_\alpha(t-a,0)$,

for all $\alpha \in$ ¡ $\setminus\{\ldots.-2,-1,0,1,2,\ldots.\}$. (3)

$$g_\alpha(t+1,a) - g_{\alpha-1}(t+1,a) = \frac{((t+1)-a)^{\overline{\alpha}}}{\Gamma(\alpha+1)} - \frac{((t+1)-a)^{\overline{\alpha-1}}}{\Gamma((\alpha-1)+1)}$$

$$= \frac{((t+1)-a)^{\overline{\alpha}}}{\Gamma(\alpha+1)} - \frac{((t+1)-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)}$$

Finally,

we get $$g_\alpha(t,a) = \frac{(t-a)^{\overline{\alpha}}}{\Gamma(\alpha+1)}.$$

IV. ALGEBRAIC PROPERTIES

A. Power Rule 4.1

Let $\alpha > 0$ and $\mu > -1$. Then,

1) $$\nabla_a^{-\alpha}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a)^{\overline{\mu+\alpha}}, \quad t \in \text{¥}_a.$$

2) $$\nabla_a^{\alpha}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(t-a)^{\overline{\mu-\alpha}}, \quad t \in \text{¥}_{a+N}.$$

B. Lemma 4.2

The following identities are valid.

1) $$\nabla t^{\overline{\alpha}} = \alpha t^{\overline{\alpha-1}},$$

2) $$t^{\overline{\alpha}}(t+\alpha)^{\overline{\beta}} = t^{\overline{\alpha+\beta}},$$

3) $$\nabla_1^{-\upsilon} t^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\upsilon+1)} t^{\overline{\mu+\upsilon}},$$

4) $$\nabla_0^{-\upsilon}(t+1)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\upsilon+1)}(t+1)^{\overline{\mu+\upsilon}}$$

5) $$\sum_{n=0}^{m} \frac{t^{\overline{n}}}{\Gamma(n+1)} = \frac{(t+1)^{\overline{m}}}{\Gamma(m+1)},$$

6) $$\nabla_a^{-(\mu)} f(t)|_{t=a} = f(a),$$

if $0 < \mu$.

$$\nabla_a^{-(\mu)} f(t)|_{t=a+1} = \mu f(a) + f(a+1) \text{ if } 0 < \mu.$$

1) Proof

1) are proved by definition of gamma function identities.
2) Consider,

$$t^{\overline{\alpha}}(t+\alpha)^{\overline{\beta}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}.\frac{\Gamma(t+\alpha+\beta)}{\Gamma(t+\alpha)}$$

$$= \frac{\Gamma(t+\alpha+\beta)}{\Gamma(t)}$$

$$t^{\overline{\alpha}}(t+\alpha)^{\overline{\beta}} = t^{\overline{\alpha+\beta}}.$$

3) This statement was followed from the power rule.

We know that, the power rule

$$\nabla_a^{-\alpha}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a)^{\overline{\mu+\alpha}},$$

Which implies that, $\nabla_1^{-\upsilon} t^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\upsilon+1)} t^{\overline{\mu+\upsilon}}$.

4) Similarly, we prove the statement was followed from the power rule.

We know that, the power rule

$$\nabla_a^{-\alpha}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a)^{\overline{\mu+\alpha}},$$

Which implies that, $\nabla_0^{-\upsilon}(t+1)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\upsilon+1)}(t+1)^{\overline{\mu+\upsilon}}$

.

5) By using induction method on m and pascal's triangle.

Since $\frac{(t+1)^{\overline{m}}}{\Gamma(m+1)} = \begin{pmatrix} t+m \\ t \end{pmatrix}$.

Thus, by induction on m,

$$\sum_{n=0}^{k+1} \frac{t^{\overline{n}}}{\Gamma(n+1)} = \begin{pmatrix} t+k \\ t \end{pmatrix} + \frac{t^{\overline{k+1}}}{\Gamma(k+2)}$$

$$\sum_{n=0}^{k+1} \frac{t^{\overline{n}}}{\Gamma(n+1)} = \begin{pmatrix} t+k \\ t \end{pmatrix} + \begin{pmatrix} t+k \\ t-1 \end{pmatrix} = \begin{pmatrix} t+k+1 \\ t \end{pmatrix}.$$

6) Already know that, $\nabla_a^{-\mu} f(t) = \sum_{s=a}^{t} \frac{(t-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)} f(s)$.

where $\rho(s) = s-1$.

Now substitute, $t = a$ and we get

$$\nabla_a^{-\mu} f(t)|_{t=a} = \sum_{s=a} \frac{(a-(s-1))^{\overline{\mu-1}}}{\Gamma(\mu)} f(s).$$

Which implies $\nabla_a^{-(\mu)} f(t)|_{t=a} = f(a)$.

7) Similarly, we know that

$$\nabla_a^{-\mu} f(t) = \sum_{s=a}^{t} \frac{(t-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)} f(s).$$

where $\rho(s) = s-1$.

Now substitute, $t = a+1$ and we get

$$\nabla_a^{-\mu} f(t)|_{t=a+1} = \sum_{s=a}^{a+1} \frac{(a-(s-1))^{\overline{\mu-1}}}{\Gamma(\mu)} f(s).$$

Which implies $\nabla_a^{-(\mu)} f(t)|_{t=a+1} = \mu f(a) + f(a+1)$

C. Result 4.3

For any $\upsilon \in$ ¡ $\setminus\{\ldots,-2,-1,0\}$.

1) $N_1(t^{\overline{\upsilon-1}})(s) = \frac{\Gamma(\upsilon)}{s^{\upsilon}}$, $|1-s| < 1$.

2) $N_1(t^{\overline{\upsilon-1}}\alpha^{-t})(s) = \frac{\alpha^{\upsilon-1}\Gamma(\upsilon)}{(s+\alpha-1)^{\upsilon}}$, $|1-s| < \alpha$.

3) $N_1(t^{\overline{\upsilon}})(s) = \frac{\upsilon}{s} N_1(t^{\overline{\upsilon-1}})$.

4) $N_a f(t+1) = (1-s)^{-1} N_{a+1} f(t)$.

5) $AN_1 f(t) = -\frac{Af(0)}{1-s} + AN_0 f(t)$.

6) $N_a\left(\nabla_a^{-\upsilon} f(t)\right) = s^{-\upsilon} N_a(f(t))(s)$.

7) If $0 < \upsilon \le 1$,
$N_{a+1}\left(\nabla_a^{\upsilon} f(t)\right)(s) = s^{\upsilon} N_a(f(t))(s) - (1-s)^{a-1} f(a)$

8) $N_{a+2}(\nabla^2 f(t))(s) = s^2 N_a(f(t))(s) - s(1-s)^{a-1} f(a)$

9) $-(1-s)^a \nabla f(a+1)$

10) If $0 < \upsilon \le 2$,

$$\begin{aligned} N_{a+2}(\nabla_a^{\upsilon} f(t))(s) &= s^{\upsilon} N_a(f(t))(s) \\ &-s(1-s)^{a-1} f(a) - (1-s)^a \nabla_a^{\upsilon-1} f(a+1) \\ &= s^{\upsilon} N_a(f(t))(s) - s(1-s)^{a-1} f(a) \\ &-(1-s)^a (f(a+1) - (\upsilon-1) f(a)) \end{aligned}$$

11) $N_0\left(\left(\frac{1}{1-a^2}\right)^{t+1}\right) = \frac{1}{(1-s)(s-a^2)}$.

D. Lemma 4.4

For any $\upsilon \in$ ¡ $\setminus\{\ldots,-2,-1,0\}$.

1) $\frac{1}{s^{\upsilon}+A} = \sum_{n=0}^{\infty} (-1)^n A^n s^{-\upsilon(n+1)}$

2) $= \sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma(\upsilon(n+1))} N_1(t^{\overline{\upsilon(n+1)-1}})$

3) $\frac{1}{s(s^{\upsilon}+A)} = \sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma(\upsilon(n+1)+1)} N_1(t^{\overline{\upsilon(n+1)}})$

4) $\frac{1}{(1-s)(s^{\upsilon}+A)} = \sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma(\upsilon(n+1))} N_0((t+1)^{\overline{\upsilon(n+1-1)}})$

1) *Proof*

Then the above three conditions are followed from previous result statements.

Then (i) is obviously true.

In particular, (ii)

$$\frac{1}{s(s^{\upsilon}+A)} = \frac{1}{s}\left[\sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma(\upsilon(n+1))} N_1((t+1)^{\overline{\upsilon(n+1-1)}})\right]$$

by (i)

$$= \sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma(\upsilon(n+1))} \frac{1}{s} N_1((t+1)^{\overline{\upsilon(n+1-1)}})$$

by using previous result, we get

$$\frac{1}{s(s^{\upsilon}+A)} = \sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma(\upsilon(n+1)+1)} N_1((t+1)^{\overline{\upsilon(n+1-1+1)}})$$

$$\frac{1}{s(s^{\upsilon}+A)}=\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma(\upsilon(n+1)+1)}N_1((t+1)^{\overline{\upsilon(n+1)}}).$$

Now (iii) $\dfrac{1}{(1-s)(s^{\upsilon}+A)}$

$$=\frac{1}{(1-s)}\left[\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma(\upsilon(n+1))}N_1((t+1)^{\overline{\upsilon(n+1)-1}})\right]$$

$$=\left[\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma(\upsilon(n+1))}\frac{1}{(1-s)}N_1((t+1)^{\overline{\upsilon(n+1)-1}})\right]$$

$$=\left[\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma(\upsilon(n+1))}(1-s)^{-1}N_1((t+1)^{\overline{\upsilon(n+1)-1}})\right]$$

We already know that, $N_a f(t+1)=(1-s)^{-1}N_{a+1}f(t)$

Which implies that

$$\frac{1}{(1-s)(s^{\upsilon}+A)}=\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma(\upsilon(n+1))}N_0((t+1)^{\overline{\upsilon(n+1)-1}}).$$

V. Two-Term Linear Fractional Nabla Difference Equations

An initial value problem for a two-term linear equation of the form

$$\nabla_0^{\upsilon}f(t)+Af(t)=0,\ f(0)=a,\ 0<\upsilon\le 1,\quad t=1,2,3,\ldots\ (1)$$

Extended to an initial value problem for a two-term, linear, nonhomogeneous equation is

$$\nabla_0^{\upsilon}f(t)+Af(t)=k,\ f(0)=a,\ 0<\upsilon\le 1,\quad t=1,2,3,\ldots\ (2)$$

And is demonstrated here.

Applying the N_1-transform on both side of the equation (2) to obtain

$$N_1(\nabla_0^{\upsilon}f(t))+N_1Af(t))=N_1k$$

by result 4.3 (vii), (v)and (i) we get

$$s^{\upsilon}N_0f(t)-(1-s)^{-1}f(a)-\frac{Af(0)}{1-s}+AN_0f(t)=\frac{k}{s}$$

$$(s^{\upsilon}+A)N_0f(t)-\frac{1}{(1-s)}a-\frac{Aa}{1-s}=\frac{k}{s}$$

$$(s^{\upsilon}+A)N_0f(t)-\frac{(1+A)}{(1-s)}a=\frac{k}{s}$$

$$(s^{\upsilon}+A)N_0f(t)=\frac{k}{s}+\frac{(1+A)}{(1-s)}a$$

$$N_0f(t)=\frac{1}{(s^{\upsilon}+A)}\left[\frac{k}{s}+\frac{(1+A)}{(1-s)}a\right]\quad(3)$$

$$N_0f(t)=\frac{k}{s(s^{\upsilon}+A)}+\frac{(1+A)}{(1-s)(s^{\upsilon}+A)}a.$$

By using lemma 4.4 (ii) and (iii), to find $f(t)$,

$$N_0f(t)=\frac{k}{s(s^{\upsilon}+A)}+\frac{(1+A)}{(1-s)(s^{\upsilon}+A)}a$$

$$=k\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma(\upsilon(n+1)+1)}N_1(t^{\overline{\upsilon(n+1)}})$$

$$+(1+A)a\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma(\upsilon(n+1))}N_0((t+1)^{\overline{\upsilon(n+1)-1}})\quad(4)$$

$$N_0f(t)=k\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma(\upsilon(n+1)+1)}N_0(t^{\overline{\upsilon(n+1)}})$$

$$+(1+A)a\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma(\upsilon(n+1))}N_0((t+1)^{\overline{\upsilon(n+1)-1}})$$

Note, we declare $N_1(t^{\overline{\upsilon(n+1)}})=N_0(t^{\overline{\upsilon(n+1)}})$. We have (4) implies that

$$f(t)=k\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma(\upsilon(n+1)+1)}(t^{\overline{\upsilon(n+1)}})\quad(5)$$

$$+(1+A)a\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma(\upsilon(n+1))}((t+1)^{\overline{\upsilon(n+1)-1}})$$

This is a solution of equation (2) and the asymptotic property is,

$$\lim_{k\to\infty}\frac{\Gamma(k+\alpha)}{k^{\alpha}\Gamma(k)}=1,\quad(6)$$

for all $\alpha\in$ ¡ ,

Consider the term

$$\sum_{n=0}^{\infty}\frac{(-1)^n A^n}{\Gamma(\upsilon(n+1)+1)}(t^{\overline{\upsilon(n+1)}})$$

And consider the limit of the ratio

$$\lim_{n\to\infty}\frac{t^{\overline{\upsilon(n+2)}}}{\Gamma(\upsilon(n+2)+1)}\Big/\frac{t^{\overline{\upsilon(n+1)}}}{\Gamma(\upsilon(n+1)+1)}\quad(7)$$

Now (7), first consider the term $\dfrac{t^{\overline{\upsilon(n+2)}}}{\Gamma(\upsilon(n+2)+1)}\Big/\dfrac{t^{\overline{\upsilon(n+1)}}}{\Gamma(\upsilon(n+1)+1)}$ and rewrite the term, we get

$$\frac{t^{\overline{\upsilon(n+2)}}}{\Gamma(\upsilon(n+2)+1)}\frac{\Gamma(\upsilon(n+1)+1)}{t^{\overline{\upsilon(n+1)}}}$$

$$=\frac{\Gamma(t+\upsilon(n+2))}{\Gamma t\Gamma(\upsilon(n+2)+1)}\frac{\Gamma(\upsilon(n+1)+1)\Gamma t}{\Gamma(t+\upsilon(n+2))}$$

$$=\frac{\Gamma(t+\upsilon(n+2))}{\Gamma(\upsilon(n+2)+1)}\frac{\Gamma(\upsilon(n+1)+1)}{\Gamma(t+\upsilon(n+1))}$$

$$\left(\frac{(\upsilon(n+1)+1)^{\upsilon}(t+\upsilon(n+1))^{\upsilon}}{(\upsilon(n+1)+1)^{\upsilon}(t+\upsilon(n+1))^{\upsilon}}\right)$$

$$=\frac{\Gamma(t+\upsilon(n+2))}{(t+\upsilon(n+1))^{\upsilon}\Gamma(t+\upsilon(n+1))}$$

$$\frac{(\upsilon(n+1)+1)^{\upsilon}\Gamma(\upsilon(n+1)+1)}{\Gamma(\upsilon(n+2)+1)}\left(\frac{(t+\upsilon(n+1))^{\upsilon}}{(\upsilon(n+1)+1)^{\upsilon}}\right)$$

Apply limit of the ratio

$$\frac{\Gamma(t+\upsilon(n+2))}{(t+\upsilon(n+1))^{\upsilon}\Gamma(t+\upsilon(n+1))}$$

And

$$\frac{(\upsilon(n+1)+1)^{\upsilon}\Gamma(\upsilon(n+1)+1)}{\Gamma(\upsilon(n+2)+1)}$$

Here $k = t + \upsilon(n+1)$, $\alpha = \upsilon$, and $k = \upsilon(n+1)+1$, $\alpha = \upsilon$ respectively.
Then

$$\lim_{n\to\infty} \frac{t^{\overline{\upsilon(n+2)}}}{\Gamma(\upsilon(n+2)+1)} \frac{\Gamma(\upsilon(n+1)+1)}{t^{\overline{\upsilon(n+1)}}} = 1.$$

So, we can apply the ratio test and the convergences reduces to the condition $|A| < 1$. The proof is valid at the order of υ

A. Example 5.1

Let as consider an initial value problem

$$\nabla_0^{\upsilon} f(t) + Af(t) = k,\ f(0) = f_0, \qquad f(1) = f_1,$$

$$1 < \upsilon \le 2, \qquad t = 2,3,4,\dots. \quad (1)$$

Applying N_2 -transform on both side we get,

$$N_2(\nabla_0^{\upsilon} f(t)) + AN_2 f(t) = N_2 k$$

$$N_2(\nabla_0^{\upsilon} f(t)) + AN_2 f(t) + Aa_1 - Aa_1 + A\frac{a_0}{1-s} - A\frac{a_0}{1-s} = N_2 k$$

$$N_2(\nabla_0^{\upsilon} f(t)) + AN_2 f(t) + A\left(f(1) + \frac{f(0)}{1-s} \right)$$

$$-A\left(a_1 - \frac{a_0}{1-s} \right) = N_2 k \qquad \text{By (1)}$$

$$N_2(\nabla_0^{\upsilon} f(t)) + AN_0 f(t)$$

$$-\left(Aa_1 - A\frac{a_0}{1-s} \right) = k\left(\frac{1-s}{s} \right)$$

By using result 4.3 (ix) to obtain

$$(s^{\upsilon} + A)N_0 f(t) - \frac{a_0 s}{1-s} - (a_1 - (\upsilon-1)a_0)$$

$$-Aa_1 - \frac{A}{1-s}a_0 = k\left(\frac{1-s}{s} \right)$$

$$N_0 f(t) = \frac{k(1-s)}{s(s^{\upsilon}+A)} + \frac{a_0 s}{(1-s)(s^{\upsilon}+A)}$$

$$+\frac{(a_1 - (\upsilon-1)a_0)}{(s^{\upsilon}+A)} + \frac{Aa_1}{(s^{\upsilon}+A)} + \frac{Aa_0}{(1-s)(s^{\upsilon}+A)}$$

Apply lemma 4.4, we get

$$N_0 f(t) = k\sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma(\upsilon(n+1)+1)} N_1(t^{\overline{\upsilon(n+1)}})$$

$$-k\sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma(\upsilon(n+1))} N_1(t^{\overline{\upsilon(n+1)-1}})$$

$$+\frac{a_0}{1-s}\sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma(\upsilon(n+1)-1)} N_1(t^{\overline{\upsilon(n+1)-2}})$$

$$+\frac{Aa_0}{1-s}\sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma(\upsilon(n+1))} N_1(t^{\overline{\upsilon(n+1)-1}})$$

$$+((1+A)a_1 - (\upsilon-1)a_0)\sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma(\upsilon(n+1))} N_1(t^{\overline{\upsilon(n+1)-1}}).$$

By using result 4.3(v) and $\frac{1}{\Gamma 0} = 0$ in the first, second and final terms, merge the second and final terms, and in the third and fourth terms, apply result 4.3(iv) to obtain

$$N_0\sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma(\upsilon(n+1)-1)} N_0((t+1)^{\overline{\upsilon(n+1)-2}})$$

$$+a_0\sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma(\upsilon(n+1)-1)} N_0((t+1)^{\overline{\upsilon(n+1)-2}})$$

$$+Aa_0\sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma(\upsilon(n+1))} N_0((t+1)^{\overline{\upsilon(n+1)-1}})$$

$$+((1+A)a_1 - (\upsilon-1)a_0 - k)\sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma(\upsilon(n+1))} N_0(t^{\overline{\upsilon(n+1)-1}}).$$

Thus,

$$f(t) = k\sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma(\upsilon(n+1)+1)} (t^{\overline{\upsilon(n+1)}})$$

$$+a_0\sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma(\upsilon(n+1)-1)} ((t+1)^{\overline{\upsilon(n+1)-2}})$$

$$+Aa_0\sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma(\upsilon(n+1))} ((t+1)^{\overline{\upsilon(n+1)-1}})$$

$$+((1+A)a_1 - (\upsilon-1)a_0 - k)\sum_{n=0}^{\infty} \frac{(-1)^n A^n}{\Gamma(\upsilon(n+1))} (t^{\overline{\upsilon(n+1)-1}})$$

Provides a solution of (1) and the ratio test imply that absolutely converges if $|A| < 1$, for $t = 0,1,2,..$

VI. Conclusion

In this article the Riemann-Liouville Type and Caputo Type Fractional Nabla Difference Equations has been discussed. A new generalized Two-term linear Fractional Nabla Difference Equation viz. theorem and example are investigated. We have also obtained the taylors monomials for any function $f(t)$.

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