A Singular Initial-Value Problem for Third-Order Differential Equations

G. Pushpalatha[1] S. K. Reka[2]
[1]Assistant Professor [2]Research Scholar
[1,2]Department of Mathematics
[1,2]Vivekanandha College of Arts and Sciences for Women, Tiruchengode, Namakkal, Tamilnadu, India

Abstract— This open article explored the existence of solution to initial-value problems for third-order nonlinear singular differential equations. The existence of solution exaggerated the terms of simple initial-value problem. Local existence and uniqueness of solutions are demonstrated below the state of dealing which are feeble than former knowing state.

Key words: Emden-Fowler Equations, Ascoli-Arzelalemma, Caratheodory Conditions, Lipschitz Condition

I. Introduction

In modern age, the research of singular initial-value problems (IVP's) of the type

$$w'' + 2t^{-1}w' + w^n(t) = 0$$

$$w(0) = 1, \qquad w'(0) = 0, \qquad (1)$$

have specified the study of more mathematicians and physicists . It is the main objective of this paper to deal the more general IVPs of the form

$$w'''(t) + p(t)w''(t) + q\big(t, w(t)\big)w' + r(t, w(t), w'(t)) + \chi(t) = 0,$$

$$w(0) = a,\ w'(0) = b,\ w''(0) = c, \quad t > 0, \qquad (2)$$

and to make additional process beyond the attainments made so far in this concern. The case $q = f(t)g(t)$ represents to Emden-Flower equations.

The function $p(t)$ in (2) may be singular at $t = 0$. Note that the problem (2) enlarged some well-known IVPs in the written works.

In the case $b = 0$ the existence of the problem (2) has been studied in [2], where the authors shown the essential of the state $b = 0$ for the existence. We search the conditions for $p(t)$,$q(t, w(t)$ and $r(t, w(t), w'(t))$ to assurance to the existence of the solution for $b \neq 0$.

II. Existence Theorems

We say that $w(t)$ is a solution to (2) if and only if there exists some $T > 0$ such that

1) $w(t)$and $w'(t)$ are absolutely continuous on $[0, T]$,
2) $w(t)$ satisfies the equation gives in (2) almost everywhere on $[0.T]$,
3) $w(t)$ Satisfies the initial conditions gives in (2).

In this section, we generalize the existence theorem of solutions.

A. Theorem 1

Let p and q satisfy the following conditions:

(R1) p is measurable on $[0, T]$;
(R2) $p \geq 0$;
(R3) $\int_0^1 sp(s)ds < \infty$;
(R4) there exist η, γ with $\eta < a < \gamma$ with $k > 0$ such that
(a) For each $t \in (0,1], q(t,\cdot)$ is continuous on $[\eta, \gamma]$;
(b) For each $w \in [\eta, \gamma], q(\cdot, w)$ is measurable on [0,1];
(c) $|q(t, w)| \leq K$.

Then a solution to the initial-value problem (2) with $b = 0$ exists.

In [5] the authors demonstrated the importance of the condition $b = 0$ for the existence.

To overcome the difficulties in the case $b \neq 0$ we consider a generalization of Theorem 1 and show that the statement of the theorem is true without condition (R3) and with weaker conditions on $q(t, w)$.

B. Theorem 2

Suppose that $p(t)$ is integrable on the interval $[e, f]$ for all $e > 0$ and p and q satisfy the following conditions:

(R1) p is measurable on $[0, T]$;
(R2) $p \geq 0$;
$(R4^*)$ there exist η, γ with $\eta < a < \gamma$ with $k > 0$ and an integrable (improper, in general) $\chi(t)$ such that

1) for each $t \in (0,1], q(t,\cdot)$ is continuous on $[\eta, \gamma]$;
2) for each $w \in [\eta, \gamma], q(\cdot, w)$ is measurable on [0,1];
3) $|q(t, w) - \chi(t)| \leq K$.

Then a solution to the initial-value problem (2) exists for all $b \in R$ such that

$$b = z'(0), \qquad (3)$$

Where $\mathcal{Z}(t) \in C[0,1]$ is a solution of the problem

$$z'''(t) + p(t)\mathcal{Z}'' + q\big(t, w(t)\big)\mathcal{Z}' + \chi(t) = 0$$

$$\mathcal{Z}(0) = a,\ \mathcal{Z}'^{(0)} = b,\ \mathcal{Z}''(0) = c,\ t > 0. \ (4)$$

That is we suppose the existence of solution of the problem (4) for some $\chi(t)$. For the problems with $b = 0$, the initial-value problem (4) always has a solution $\mathcal{Z}(t) = a$, for $\chi(t) = 0$. so Theorem 1 corresponds to the case $\chi(t) = 0$ and $\mathcal{Z}(t) = a$.

One of the advantages of Theorem 2 is that the problem (4) always have a solution for some appropriate $\chi(t)$; for example, for $\chi(t) = -bp(t) - cq(t, w(t))$, the problem (4) has a solution $\mathcal{Z}(t) = a + bt + ct^2$. The conclusion of the theorem remains valid for all solutions of (4).

It is clear from the conclusion of the Theorem 2 that the interval [0, 1] can be taken as $[0, t_0]$ for some small enough $t_0 > 0$.

1) Proof of Theorem 2

For $t \in (0,1]$, we explain the functions

$$m(t) \equiv \left(\int_1^t p(s)\, ds\right) \geq 0,$$

$$m_1(t) = \left(-\int_1^t p(s)\, ds\right),$$

$$N(t) = \int_1^t h_1(s)\, ds. \qquad (5)$$

The function $m(t)$ is a bounded function which is continuous for $t \in (0,1]$. It is continuous or has a elimination discontinuity at $t = 0$ and is differentiable almost everywhere.

In the problem of (2) is might be equal for upcoming integral equation:

$w(t) = \int_0^t \left(N(s)e^{\int_1^s p(\tau)d\tau} - N(t)e^{\int_1^s p(\tau)d\tau} - N(u)e^{\int_1^s p(\tau)d\tau}\right) \times$
$[q(s,w(s)) - \chi(s)]ds + \mathcal{Z}(t).$
$= \int_0^t \left(N(s)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds - \int_0^t \left(N(t)e^{\int_1^s p(\tau)d\tau}\right)$
$[q(s,w(s)) - \chi(s)]ds - \int_0^t \left(N(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds + \mathcal{Z}(t).$
$= \left|\int_0^t \left(N(s)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds\right| - \left|\int_0^t \left(N(t)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds\right| - \left|\int_0^t \left(N(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds\right| + |\mathcal{Z}(t)|.$

First, let us exhibit the existence of the integral in (6). We have for any $\delta > 0$ that

$\left|\int_0^t \left(N(s)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds\right|$, here $[q(s,w(s)) - \chi(s)] \le K$

$\le K \left|\int_0^t \left(N(s)e^{\int_1^s p(\tau)d\tau}\right)\right|$

$$= \mathrm{K}\left|\int_\delta^t \int_1^s m_1(u)e^{\int_1^s p(\tau)d\tau} du\, ds\right| \quad (7)$$

$= \mathrm{K}\left|\int_\delta^t \int_1^s e^{-\int_1^u p(v)dv} e^{\int_1^s p(\tau)d\tau} du\, ds\right|$

It purse on from $u \ge s$ on the set $[s,1] \times [0,t]$ that .

$$\int_\delta^t \int_1^s e^{-\int_1^u p(v)dv} e^{\int_1^s p(\tau)d\tau} = e^{-\int_s^u p(v)dv} \le 1 \quad (8)$$

$$\left|\int_0^t \left(N(s)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds\right| \le K\left|t - \frac{t^2}{2}\right|. \quad (9)$$

In similar way we recieved

$\left|\int_0^t \left(N(t)e^{\int_1^t p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds\right| \le K \left|\int_0^t \left(N(t)e^{\int_1^t p(\tau)d\tau}\right)\right|$

$= \mathrm{K}\left|\int_\delta^t \int_1^t m_1(u)e^{\int_1^t p(\tau)d\tau} du\, ds\right|$

$= \mathrm{K}\left|\int_\delta^t \int_1^t e^{-\int_1^u p(v)dv} e^{\int_1^t p(\tau)d\tau} du\, ds\right|.$

It follows from $u \ge s$ on the set $[s,1] \times [0,t]$ that

$\int_\delta^t \int_1^t e^{-\int_1^u p(v)dv} e^{\int_1^t p(\tau)d\tau} = e^{-\int_t^u p(v)dv} \le 1$

$$\left|\int_0^t \left(N(t)e^{\int_1^t p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds\right| \le K\,|t - t^2|. \quad (10)$$

In like manner we obtain

$\left|\int_0^t \left(N(u)e^{\int_1^u p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds\right| \le K \left|\int_0^t \left(N(u)e^{\int_1^u p(\tau)d\tau}\right)\right|$

$= \mathrm{K}\left|\int_\delta^t \int_1^u m_1(u)e^{\int_1^u p(\tau)d\tau} du\, ds\right|$

$= \mathrm{K}\left|\int_\delta^t \int_1^u e^{-\int_1^u p(v)dv} e^{\int_1^u p(\tau)d\tau} du\, ds\right|.$

It follows from $u \ge s$ on the set $[s,1] \times [0,t]$ that

$\int_\delta^t \int_1^u e^{-\int_1^u p(v)dv} e^{\int_1^u p(\tau)d\tau}$
$= e^{-\int_u^u p(v)dv} \le 1$,

$$\left|\int_0^t \left(N(s)e^{\int_1^u p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds\right| \le K\,|\,t - 2t^2|. \quad (11)$$

So the right-hand side of (6) label the outcome for any $p(t) \ge 0$ for $|q(t,w) - \chi(t)| \le K$ and

$$\lim_{\delta\to 0}\int_\delta^t \left(N(s)e^{\int_1^s p(\tau)d\tau} - N(t)e^{\int_1^s p(\tau)d\tau} - N(u)e^{\int_1^s p(\tau)d\tau}\right) \times [q(s,w(s)) - \chi(s)]ds + \mathcal{Z}(t) \quad (12)$$

$= \int_0^t \left(N(s)e^{\int_1^s p(\tau)d\tau} - N(t)e^{\int_1^s p(\tau)d\tau} - N(u)e^{\int_1^s p(\tau)d\tau}\right) \times [q(s,w(s)) - \chi(s)]ds + \mathcal{Z}(t).$

Now let us find the derivatives $w'(t), w''(t)$ and $w'''(t)$ from (6) by using the Leibniz rule:

$w'(t) =$

$$\left(\int_0^t \left(N(s)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds - \int_0^t \left(N(t)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds - \int_0^t \left(N(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds + \mathcal{Z}(t)\right)'$$

$= \int_0^t \left(N'(s)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds + \int_0^t \left(N(s)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds - \int_0^t \left(N'(t)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds - \int_0^t \left(N(t)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds - \int_0^t \left(N'(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds - \int_0^t \left(N(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds + \mathcal{Z}'(t)$

$= N'(t)e^{\int_1^t p(\tau)d\tau}[q(t,w(t)) - \chi(t)]ds + \int_0^t \left(N(s)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds - \int_0^t \left(N'(t)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds - \int_0^t \left(N(t)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds - \int_0^t \left(N'(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds - \int_0^t \left(N(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds + \mathcal{Z}'(t)$

$= N'(t)e^{\int_1^t p(\tau)d\tau}[q(t,w(t)) - \chi(t)]ds + \int_0^t \left(N(s)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds - \left(N'(t)e^{\int_1^t p(\tau)d\tau}\right)[q(t,w(t)) - \chi(t)]dt - \int_0^t \left(N(t)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds - \int_0^t \left(N'(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds - \int_0^t \left(N(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds + \mathcal{Z}'(t)$

$= \int_0^t \left(N(s)e^{\int_1^s p(\tau)d\tau}\right)\left[[q(s,w(s)) - \chi(s)]dt\right] - \int_0^t \left(N(t)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds - \int_0^t \left(N'(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds - \left[\left(N(u)e^{\int_1^t p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds\right] + \mathcal{Z}'(t)$

$$w'(t) = \int_0^t \left(N(s)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds - \int_0^t \left(N(t)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds - \int_0^t \left(N'(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds - \int_0^t \left(N(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) - \chi(s)]ds + \mathcal{Z}'(t) \quad (13)$$

$w''(t) =$

$$\begin{pmatrix} \int_0^t \left(N(s)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s))-\chi(s)] \\ ds - \int_0^t \left(N(t)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) \\ -\chi(s)]ds - \int_0^t \left(N'(u)e^{\int_1^s p(\tau)d\tau}\right) \\ [q(s,w(s))-\chi(s)]ds - \\ \int_0^t \left(N(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s))-\chi(s)] \\ ds + \mathcal{Z}'(t) \end{pmatrix}'$$

$= \int_0^t \left(N'(s)\, e^{\int_1^s p(\tau)d\tau}\right) ds \ [q(s,w(s))-\chi(s)]ds+ \int_0^t \left(N(s)e^{\int_1^s p(\tau)d\tau}\right) [q(s,w(s))-\chi(s)]ds - \int_0^t N'(t)\left(e^{\int_1^s p(\tau)d\tau}\right) [q(s,w(s))-\chi(s)]ds - \int_0^t \left(N(t)e^{\int_1^s p(\tau)d\tau}\right) [q(s,w(s))-\chi(s)]ds - \int_0^t \left(N''(u)e^{\int_1^s p(\tau)d\tau}\right) [q(s,w(s))-\chi(s)]ds - \int_0^t \left(N'(u)e^{\int_1^s p(\tau)d\tau}\right) [q(s,w(s))-\chi(s)]ds - \int_0^t \left(N'(u)e^{\int_1^s p(\tau)d\tau}\right) [q(s,w(s))-\chi(s)]ds - \int_0^t \left(N(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s))-\chi(s)]ds + \mathcal{Z}''(t)$

$= \left(N'(t)\, e^{\int_1^t p(\tau)d\tau}\right)dt\,[q(t,w(t))-\chi(t)] + \left(N(t)e^{\int_1^t p(\tau)d\tau}\right) [q(t,w(t))-\chi(t)]dt - \left(N'(t)\, e^{\int_1^t p(\tau)d\tau}\right)dt\,[q(t,w(t))-\chi(t)] - \left(N(t)e^{\int_1^t p(\tau)d\tau}\right) [q(t,w(t))-\chi(t)]dt - \int_0^t \left(N''(u)e^{\int_1^s p(\tau)d\tau}\right) [q(s,w(s))-\chi(s)]ds - 2\int_0^t \left(N'(u)e^{\int_1^s p(\tau)d\tau}\right) [q(s,w(s))-\chi(s)]ds - \int_0^t \left(N(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s))-\chi(s)]ds + \mathcal{Z}''(t)$

$= -\int_0^t \left(N''(u)e^{\int_1^s p(\tau)d\tau}\right) [q(s,w(s))-\chi(s)]ds - 2\int_0^t \left(N'(u)e^{\int_1^s p(\tau)d\tau}\right) [q(s,w(s))-(s)]ds - \int_0^t \left(N(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s))-\chi(s)]ds + \mathcal{Z}''(t)$

$$w''(t) = -\int_0^t \left(N''(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s))-\chi(s)]ds - 2\int_0^t \left(N'(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s))-\chi(s)]ds - \int_0^t \left(N(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s))-\chi(s)]ds + \mathcal{Z}''(t) \quad (14)$$

$w'''(t) =$

$$\begin{pmatrix} -\int_0^t \left(N''(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) \\ -\chi(s)]ds - 2\int_0^t \left(N'(u)e^{\int_1^s p(\tau)d\tau}\right) \\ [q(s,w(s))-\chi(s)]ds \\ -\int_0^t \left(N(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) \\ -\chi(s)]ds + \mathcal{Z}''(t) \end{pmatrix}'$$

$= -\int_0^t \left(N'''(u)e^{\int_1^s p(\tau)d\tau}\right) [q(s,w(s))-\chi(s)]ds - \int_0^t \left(N''(u)e^{\int_1^s p(\tau)d\tau}\right) [q(s,w(s))-\chi(s)]ds - 2\int_0^t \left(N''(u)e^{\int_1^s p(\tau)d\tau}\right) [q(s,w(s))-\chi(s)]ds - 2\int_0^t \left(N'(u)e^{\int_1^s p(\tau)d\tau}\right) [q(s,w(s))\chi(s)]ds - \int_0^t \left(N'(u)e^{\int_1^s p(\tau)d\tau}\right) [q(s,w(s))-\chi(s)]ds - \int_0^t \left(N(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s))-\chi(s)]ds + \mathcal{Z}'''(t)$

$= -\int_0^t \left(N'''(u)e^{\int_1^s p(\tau)d\tau}\right) [q(s,w(s))-\chi(s)]ds - 3\int_0^t \left(N''(u)e^{\int_1^s p(\tau)d\tau}\right) [q(s,w(s))-\chi(s)]ds - 3\int_0^t \left(N'(u)e^{\int_1^s p(\tau)d\tau}\right) - \int_0^t \left(N(u)e^{\int_1^s p(\tau)d\tau}\right) [q(s,w(s)) - \chi(s)]ds + \mathcal{Z}'''(t)$

$$w'''(t) = -\int_0^t \left(N'''(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s))-\chi(s)]ds - 3\int_0^t \left(N''(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s))-\chi(s)]ds - 3\int_0^t \left(N'(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s))-\chi(s)]ds - \int_0^t \left(N(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s))-\chi(s)]ds + \mathcal{Z}'''(t) \quad (15)$$

It follows that (15) that

$w'''(t) + p(t)w''(t) + q(t,w(t))w' + r(t,w(t),w'(t))$

$$= \begin{pmatrix} -\int_0^t \left(N'''(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) \\ -\chi(s)]ds - 3\int_0^t \left(N''(u)e^{\int_1^s p(\tau)d\tau}\right) \\ [q(s,w(s))-\chi(s)]ds - \\ 3\int_0^t \left(N'(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) \\ -\chi(s)]ds - \int_0^t \left(N(u)e^{\int_1^s p(\tau)d\tau}\right) \\ [q(s,w(s))-\chi(s)]ds + \mathcal{Z}'''(t) \end{pmatrix} + p(t)\begin{pmatrix} \int_0^t \left(N(s)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s))-\chi(s)] \\ ds - \int_0^t \left(N(t)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s)) \\ -\chi(s)]ds - \int_0^t \left(N'(u)e^{\int_1^s p(\tau)d\tau}\right) \\ [q(s,w(s))-\chi(s)]ds - \\ \int_0^t \left(N(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s))-\chi(s)] \\ ds + \mathcal{Z}'(t) \end{pmatrix}$$

$$+\, q(t,w(t))\begin{pmatrix} \int_0^t \left(N(s)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s))-\chi(s)]ds - \\ \int_0^t \left(N(t)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s))-\chi(s)]ds - \\ \int_0^t \left(N(u)e^{\int_1^s p(\tau)d\tau}\right)[q(s,w(s))-\chi(s)]ds + \\ \mathcal{Z}(t) \end{pmatrix} + r(t,w(t),w'(t)) \quad (16)$$

$z'''(t) + p(t)\mathcal{Z}'' + q(t,w(t))\mathcal{Z}' + \chi(t) = 0.$

That is, the problem (2) is similar to (6). Let us define the recurrence relations

$$w_0(t) = z(t), \quad (17)$$

$$w_n(t) = \int_0^t \left(N(s)e^{\int_1^s p(\tau)d\tau} - N(t)e^{\int_1^s p(\tau)d\tau} - N(u)e^{\int_1^s p(\tau)d\tau}\right) \times [q(s,w_{n-1}(s))-\chi(s)]ds + \mathcal{Z}(t), \quad (18)$$

Where $w(t)$ is a solution of the problem (4). It follows from (9), (10), and (16) that $\eta < w_n(t) < \gamma$ for $\eta < w_{n-1}(t) < \gamma$

and for small enough $t \in [0,t_0)$. Now, for $t_1,t_2,t_3 \in [0,t_0)$, we have from (9) and (10) that

$$|w_n(t_3)-w_n(t_2)-w_n(t_1)| = \left|\int_0^t \left(N(s)e^{\int_1^s p(\tau)d\tau} - N(t)e^{\int_1^s p(\tau)d\tau} - N(u)e^{\int_1^s p(\tau)d\tau}\right)\left[q(s,w_{n-1}(s)) - \chi(s)\right]ds\right|$$

$$= \left|\int_{t_1}^{t_3}\left(N(s)e^{\int_1^s p(\tau)d\tau}\right)\left[q(s,w_{n-1}(s))-\chi(s)\right]ds\right| + \left|-\int_{t_1}^{t_3}\left(N(t)e^{\int_1^s p(\tau)d\tau}\right)\left[q(s,w_{n-1}(s))-\chi(s)\right]ds\right| + \left|-\int_{t_1}^{t_3}\left(N(u)e^{\int_1^s p(\tau)d\tau}\right)\left[q(s,w_{n-1}(s))-\chi(s)\right]ds\right|$$

$$\le K\left|t_3-\frac{t_3^2}{2}\right| + K\left|t_2-t_2^2\right| + K\left|t_1-2t_1^2\right| \le K\left|t_3\left(1-\frac{t_3}{2}\right)\right| + K|t_2(1-t_2)| + K|t_1(1-2t_1)|$$

$$\le 3K(t_3-t_2-t_1)\left(1-\frac{t_3}{2}\right)(t_2-1)(2t_1-1) \le K_1(t_3-t_2-t_1), \quad (19)$$

for some constant K_1. Thus, the sequence $w_n(t)$ is uniformly bounded and uniformly continuous and, by Ascoli-Arzela lemma, there exists a continuous $w(t)$ such that $w_{n_k}(t) \to w(t)$ uniformly on $[0,T]$, for any fixed $T \in [0,t_0)$. Without loss of generality, say $w_n(t) \to w(t)$. Then

$$w(t) = \lim_{n\to\infty}\int_0^t\left(N(s)e^{\int_1^s p(\tau)d\tau} - N(t)e^{\int_1^s p(\tau)d\tau} - N(u)e^{\int_1^s p(\tau)d\tau}\right)\times\left[q(s,w(s))-\chi(s)\right]ds + \mathcal{Z}(t) \quad (20)$$

$$= \int_0^t\left(N(s)e^{\int_1^s p(\tau)d\tau} - N(t)e^{\int_1^s p(\tau)d\tau} - N(u)e^{\int_1^s p(\tau)d\tau}\right)\times\left[q(s,w(s))-\chi(s)\right]ds + \mathcal{Z}(t),$$

Using the Lebesgue dominated convergence theorem.

2) Remark

Note that the positivity condition of the function $p(t)$ can be weakened.

The positivity of $p(t)$ has been used in the proof of theorem 2 to show the (removable) continuity of the function $m(t)$ at 0. Now assuming that the following condition holds:

(K2) $|p|$ is integrable on $[d,e]$ for any fixed $d,e \in (0,1]$, $d < e$,

And $\quad L \le \int_e^f p(s)ds < +\infty$;

For some fixed L (21)

We can prove a similar theorem.

C. Theorem 3

Suppose that $p(t)$ is integrable on the interval $[e,f]$ for all $e > 0$ and p and q satisfy the following conditions:

(R1) p is measurable on $[0,T]$;

(R2) $L \le \int_e^f p(s)ds < +\infty$; for some fixed L

($R4^*$)There exist η,γ with $\eta < a < \gamma$ with $k > 0$ and an integrable (improper, in general) $\chi(t)$ such that

1) for each $t \in (0,1]$, $q(t,\cdot)$ is continuous on $[\eta,\gamma]$;
2) for each $w \in [\eta,\gamma]$, $q(\cdot,w)$ is measurable on $[0,1]$;
3) $|q(t,w)-\chi(t)| \le K$.

Then a solution to the initial-value problem (2) exists for all $b \in R$ such that

$$b = z'(0),$$

Where $\mathcal{Z}(t) \in C[0,1]$ is a solution of the problem?

$$z'''(t) + p(t)\mathcal{Z}'' + q(t,w(t))\mathcal{Z}' + \chi(t) = 0, \quad \mathcal{Z}(0) = a,\ \mathcal{Z}'^{(0)} = b,\ \mathcal{Z}''(0) = c, t > 0.$$

1) Proof

We need to make some modifications to the proof of Theorem 2; instead of the inequality

$$e^{-\int_1^u p(v)dv}e^{\int_1^t p(\tau)d\tau} = e^{-\int_t^u p(v)dv} \le 1, \quad (22)$$

For $u \ge s$, we will have

$$e^{-\int_1^u p(v)dv}e^{\int_1^t p(\tau)d\tau} = e^{-\int_s^u p(v)dv} \le e^{-L}, \quad (23)$$

For small enough u and s.

It is worthy to note that the existence of the solution of the problems like

$$w''' + p(t)w'' + q(t,w(t))w' + r(t,w(t),w'(t)) = 0.$$

$$w''' + \left(\frac{a_1}{t^m} + \frac{a_{m-1}}{t^{m-1}} + \cdots + \frac{a_1}{t} + A(t)\right)w'' + \left(\frac{b_1}{t^m} + \frac{b_{m-1}}{t^{m-1}} + \cdots + \frac{b_1}{t} + B(t)\right)w' + r(t,w(t),w'(t)) = 0,$$

$$w(0) = a, w'(0) = b, \quad w''(0) = c, t > 0, \quad (24)$$

Follow from theorem 2, where $A(t)$ is differentiable function, $r(t,w(t),w'(t))$ satisfies the conditions there exist η,γ with $\eta < a < \gamma$ with $k > 0$ and an integrable (improper, in general) $\chi(t)$ such that

a) for each $t \in (0,1]$, $q(t,\cdot)$ is continuous on $[\eta,\gamma]$;
b) for each $w \in [\eta,\gamma]$, $q(\cdot,w)$ is measurable on $[0,1]$;
c) $|q(t,w)-\chi(t)| \le K$,

$a_1,a_2,\dots,a_m$ and $b_1,b_2,\dots,b_m$ are real constants, $a_m > 0$ and $b_m > 0$. Indeed for small enough t we have $p(t) > 0$ and therefore the hypotheses of theorem 2 and 3 are true for small enough $t \in [0,T]$; for $b = 0$ the problem (4) has a solution $z(t) = a$, and so (23) has a solution for all bounded $q(t,w(t))$ with Caratheodory conditions, but $b \neq 0$ the problem (23) has asolution for $q(t,w(t))$ with

$$\left|q(t,w(t)) + c\left(\frac{a_1}{t^m} + \frac{a_{m-1}}{t^{m-1}} + \cdots \frac{a_1}{t}\right)\left(\frac{b_1}{t^m} + \frac{b_{m-1}}{t^{m-1}} + \cdots + \frac{b_1}{t}\right)\right| < \mathrm{K}$$

In some small enough neighborhood of 0, since the corresponding problem (4) can be taken (e.g.) as

$$z''' + \left(\frac{a_1}{t^m} + \frac{a_{m-1}}{t^{m-1}} + \cdots + \frac{a_1}{t} + A(t)\right)z'' + \left(\frac{b_1}{t^m} + \frac{b_{m-1}}{t^{m-1}} + \cdots + \frac{b_1}{t} + B(t)\right)z' - c\left(\frac{a_1}{t^m} + \frac{a_{m-1}}{t^{m-1}} + \cdots + \frac{a_1}{t}\right)\left(\frac{b_1}{t^m} + \frac{b_{m-1}}{t^{m-1}} + \cdots + \frac{b_1}{t}\right) = 0, w(0) = a, \quad w'(0) = b, \quad w''(0) = c,$$

$$t > 0, \quad (25)$$

And has a solution $z(t) = a + bt + ct^2$. It is remarkable that for $b \neq 0$ the condition for $q(t,w(t))$ can be changed by using different functions for $\chi(t)$. For example, $\chi(t)$ can be taken as

$$\chi(t) = \frac{c_m}{t^m} + \frac{c_{m-2}}{t^{m-2}} + \cdots$$

$$= \left[-\frac{ca_m}{t^m} + \frac{1}{t^{m-2}}\left(\frac{ca_{m-1}^2}{a_m} - ca_{m-2}\right) + \frac{1}{t^{m-3}}\left(\frac{ca_{m-1}a_{m-2}}{a_m} - ca_{m-3}\right) + \cdots + \frac{1}{t}\left(\frac{ca_{m-1}a_2}{a_m} - ca_1\right) + \frac{ca_{m-1}a_2}{a_m} - cA(t) - \frac{ca_{m-1}}{a_m}\right]\left[-\frac{cb_m}{t^m} + \frac{1}{t^{m-2}}\left(\frac{cb_{m-1}^2}{b_m} - cb_{m-2}\right) + \frac{1}{t^{m-3}}\left(\frac{cb_{m-1}b_{m-2}}{b_m} - cb_{m-3}\right) + \cdots + \frac{1}{t}\left(\frac{cb_{m-1}b_2}{b_m} - ca_1\right) + \frac{cb_{m-1}b_2}{b_m} - cB(t) - \frac{cb_{m-1}}{b_m}\right] \quad (26)$$

And (4) as

$$z''' + \left(\frac{a_1}{t^m} + \frac{a_{m-1}}{t^{m-1}} + \cdots + \frac{a_1}{t} + A(t)\right)z'' + \left(\frac{b_1}{t^m} + \frac{b_{m-1}}{t^{m-1}} + \cdots + \frac{b_1}{t} + B(t)\right)z' + \chi(t) = 0,$$

$$w(0) = a, w'(0) = b, w''(0) = c, t > 0, \quad (27)$$

With solution

$z(t) = a + bt + ct^2 - \left(\frac{ca_{m-1}}{2a_m}\right)t^3 - \left(\frac{cb_{m-1}}{2b_m}\right)t^4.$

Continuing in like manner, the condition for $q(t, w(t))$can be reduced to

$$\left|q(t, w(t)) + \frac{ca_m}{t^m} + \frac{cb_m}{t^m}\right| < K.$$

This inequalities of the type (7)−(10) can be easily established for the function $q(t, w)$ with

$$(\text{R4} * d)\ |q(t, w(t)) - \chi(t)| \le h(t), \quad (28)$$

Where $m(t)$ absolutely integrable function, and the more general theorem is can be stated as follows.

III. EPILOGUE

We expanded the type of solvable second-order singular IVP's. we prevailing that the struggles related to the singularity can be resolve the problem of the type (2) with $p \ge 0$ or

$$L \le \int_e^f p(s)ds < +\infty; \text{ for some fixed L.}$$

The problem of the existence of a solution is decrease to the searching of a solution to some more uncomplicated problems like (4).

The method used here can be applicable for the problems on the existence of solutions of boundary value problems. The authors demonstrated the significant theorems on the existence and uniqueness of the solution of the equation

$$w'''(t) + p(t)w''(t) + q(t)w' + r(t)w + \chi(t) = 0, \quad (64)$$

In few boundary conditions, in part of an auxiliary homogeneous equation

$$w'''(t) + p(t)w''(t) + q(t)w' + r(t)w = 0. \quad (65)$$

Our method is separate from the method. We accepted the recent auxiliary (nonhomogeneous, but easily solvable) (4) which include (65).

The conditions we attain are weaker than the previously known ones and can be simply decreased to various remarkable cases.

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