A Narrative Study in Ramsey Graph Theory using Rank Coloring

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Abstract— Now a day's Graph Ramsey Theory has grown to one of the most familiar active areas. A major impetus behind the early development of Graph Ramsey Theory was the hope that it would eventually lead to methods for determining larger values of the classical Ramsey number R (m ,n).In fact, it is probably safe to say that the results arising from Graph Ramsey Theory will prove to be more valuable and interesting than knowing the exact values of R(m ,n).This paper deals with the construction of extremal graphs. it is divided into three major areas: i) Ramsey numbers with Turan's theorem ii) Rank coloring with graph. First once called Ramsey numbers. For the classical Ramsey numbers. G is taken to be a complete graph. We begin by giving Ramsey theorem which gives a recurrence relation for r(k,1), k>=2 and 1>=2.The existence of r (k1,k2…….km) is guaranteed by Ramsey's original theorem. Ramsey theorem yields an upper bound for r(k,1). Erdos's theorem (1947) yields a lower bound for r(k,k).it deals with Turan's theorem and some of its consequences. Turan's theorem gives upper bounds for the number of edges of graphs with certain specified properties.The study of Ramsey has become important since cubic graphs play a vital role in several situations in graph theory, as for instance, in the four color problem. Finally called Rank coloring graph, A ranking of a graph is a coloring of the vertex set with positive integers such that on every path connecting two vertices of the same color there is a vertex of larger color. We show that the ranking number of a directed graph is bounded by that of its longest directed path plus one, and that it can be computed in polynomial time.

Key words: Ramsey theorem, Turan's theorem, Erdos's theorem, Rank coloringuip

I. INTRODUCTION

In general, Ramsey theory deals with the guaranteed occurrence of specific structures in some part of a large arbitrary structure which has been partitioned into definitely many parts [1]. The ground field is named for Frank P. Ramsey who proved its first result, but many of its most significant contributions have come from Paul Erdos. Since then, the field has exploded. It now boasts many variations on classical Ramsey theory, and its effects are felt in widespread branches of mathematics. Here we will provide a brief introduction to graph theory in order to establish the necessary background needed to explore the basics of classical Ramsey theory. Many of the results we present served as the jumping-o_ points for entire new branches of the discipline, and this brief introduction is meant only to familiarize the reader with some key ideas and fundamental results. The majority significant topics from graph theory to consider when discussing Ramsey theory is colorings. There are two foremost types of colorings, those on the vertices of a graph and those on the edges of a graph. Our discussion of Ramsey theory will require only edge colorings, but for the sake of completeness. Ramsey theorem was proved in passing, as a means to a result about logic, but it turned out to be one of the first combinatorial results that widely attracted the attention of mathematicians. We will prove the theorem in terms of 2-colorings of the edges of complete graphs, but also discuss how the result can be viewed. In terms of the old puzzle regarding mutual acquaintances and strangers at a party.

II. NITTY-GRITTY OF GRAPH THEORY

We should instigate by first introducing some significant concepts in graph theory that will allow us to develop Ramsey theory later. First, we will establish what a graph is and some important vocabulary used in the discussion of graphs.

Definition 2.1. A graph consists of two finite sets, V and E. Each element of V is called a vertex. The vertex set of a graph G is denoted by V (G) or simply V . The elements of E are unordered pairs of vertices called edges. An edge connecting vertices u and v is denoted uv and u and v are said to be the edge's ends. The edge set is denoted by E(G) or simply E. [2]

Instead of referring to a graph by it vertex and edge sets, more commonly we consider a visual depiction of a given graph. By convention, each element of the vertex set is represented by a dot, and each element of the edge set is represented by a line connecting two dots. The graphs are generally divided into two main classes, simple graphs and multigraphs. The classifications depend the presence or absence of two features, loops and multiple edges.

Definition 2.2. An edge for which the two ends are the same is called a loop at the common vertex. A set of two or more edges of a graph is called a set of multiple edges if they have the same ends. [3]

III. RAMSEY THEORY

Ramsey theory got its start and its name when Frank Ramsey published his paper on a Problem of Formal Logic. The theorem was proved in passing, as a means to a result about logic, but it turned out to be one of the first combinatorial results that widely attracted the attention of mathematicians. We will prove the theorem in terms of 2-colorings of the edges of complete graphs, but also discuss how the result can be viewed in terms of the old puzzle regarding mutual acquaintances and strangers at a party.

A. Ramsey's Theorem

Theorem 3.1. Ramsey's Theorem Given any positive integers p and q, there exists a smallest integer n = R(p; q) such that every 2-coloring of the edges of Kn contains either a complete subgraph on p vertices, all of whose edges are in color 1, or a complete subgraph on q vertices, all of whose edges are in color 2.

Proof. We will proceed by induction on p + q.

First we consider the base case in which p + q = 2. The only way this can be true is if p = q = 1, and it is clear that R(1; 1) = 1.Now we assume that the theorem holds whenever p + q < N, for some positive integer N. Let P and Q be integers such that P + Q = N. Then P + Q -1 < N, so by our assumption we know that R(P = 1;Q) and R(P;Q = 1) exist.

Consider any coloring of the edges of Kv in two colors c1 and c2, where v _ R(P -1;Q) + R(P;Q - 1). Let x be a vertex of Kv. By the pigeonhole principle and because v _ R(P -1;Q)+R(P;Q-1), we know that of the v-1 edges that x is incident to, eitherR(P - 1;Q) edges are in color c1 or R(P;Q - 1) edges are in color c2.

If x is incident to R(P -1;Q) edges of color c1, consider the KR(P-1;Q) whose vertices are the vertices joined to x by edges of color c1, that is the subgraph induced by the neighborhood of x. Because we know that R(P-1;Q) exists, there are two possible cases to consider. One is that this graph contains a KP-1 with all edges in color c1, in which case this KP-1 together with x forms a monochromatic KP in color c1. The other possibilityis that KR(P-1;Q) contains a KQ with all edges in color c2. In either case, we can see that R(P;Q) exists.

If we consider this problem in terms of people at a party, Ramsey's Theorem guarantees that there is some smallest number of people at the party required to ensure that there is either a set of p mutual acquaintances or q mutual strangers. Thus, the old puzzle that asks us to prove that with any six people at a party, among them there must be a set of three mutual aquaintances or a set of three mutual strangers actually requires us to show that R(3; 3) = 6.

We should also note that Ramsey's Theorem can be generalized to account for colorings in any finite number of colors, not just 2-colorings.

Ramsey's Theorem guarantees that this smallest integer R(p; q) exists but does little to help us determine what its value is, given some positive integers, p and q. In general, this is actually an exceedingly difficult problem.

IV. RAMSEY NUMBERS

Definition 4.1. The integers R(p; q) are known as classical Ramsey numbers.

Paul Erdos was a Hungarian mathematician who made huge contributions to the fields of combinatorics and graph theory. Commenting on the difficulty of determining Ramsey numbers, he said,Suppose an evil alien would tell mankind `Either you tell me [the value of R(5, 5)] or I will exterminate the human race.' ... It would be best in this case to try to compute it, both by mathematics and with a computer. If him before he destroys us, because we couldn't. [6]" Indeed, at the present relatively few nontrivial Ramsey numbers have been discovered.

It follows from the definition of Ramsey's Theorem that for positive integers p and q, R(p; q) = R(q; p), and we have already noted that R(1; 1) = 1. Similarly, it is easy to see that for every positive integer k, R(1; k) = 1. Determining the values of R(2; k) is only slightly more difficult.

V. RANK COLORINGS

One of the most significant topics from graph theory to consider when discussing Ramsey theory is colorings. There are two main types of colorings, those on the vertices of a graph and those on the edges of a graph. Our discussion of Ramsey theory will require only edge colorings, but for the sake of completeness we will define both types.

Definition 5.1. Given a graph G, a k-coloring of the vertices of G is a partition of V (G) into k sets C1;C2; : : : ;Ck such that for all i, no pair of vertices from Ci are adjacent.If such a partition exists, G is said to be k-colorable. [4]

Definition 5.2. Given a graph G, a k-coloring of the edges of G is any assignment of one of k colors to each of the edges of G. [4]

In our discussion of Ramsey theory, we will covenant primarily with 2-colorings of the edges of graphs. By convention, the colors referred to are typically red and blue. An example of a graph and several 2-colorings of its edges.

VI. TURAN'S THEOREM

Turan's theorem determines the maximum number of edges that a simple graph on v vertices can have containing no chique of size m+1.Turan's theorem has become the basis of a significant branch of graph theory known as external graph theory.

Extremal problems are problems that ask:

What is the largest or smallest graph with a specified property? Usually we shall use the term"largest" to mean the graph with the maximum number of edges given the number of vertices. The number of vertices in a graph G is called the order of G. The number of edges in G is called the size of G. An induced sub graph of a graph G is a sub graph of G obtained by taking a subset W of the vertices of G together with every edge of G that has both endpoints in W.

Problem:6.1. Find a largest graph G with n vertices and chromatic number two.

Since G has chromatic number two,the vertices are colored by two colors,red and blue.If there is a red vertex that is not adjacent to a blue vertex,we can add this edge and increase the number of edges.Thus every blue vertex is adjacent to every red vertex and we have a complete bipartic graph.Suppose there are m1 blue and m2 red vertices,with m1+m2=n.we shall show that m1 and m2 are as close to equal as possible,which means that they are either equal or only one apart.In other words,m1-m2≤1.

Suppose that m2≤m1.If m2 +2≤m1,then we choose another complete bipartite graph G with m2+1 red vertices and m1-1 blue vertices.Then G has m1,m2 edges and G has(m1-1)(m2+1) edges.But then we have

(m1-1)(m2+1)-m1m2=m1m2+m1-m2-1-m1m2 m1-m2-1

Since we assumed that m2≤m1-2,m1-m2-1≥1,and thus G has at least one more edge than G and is still colorable by two colors and has n vetices.This shows that m1 and m2 differ by at most one.Reffering to the first problem,where n=2h,it means m1 and m2 are both equal to h.

Problem:6.2.Find the largest graph G with n vertices and chromatic number k.

Since G is k-chromatic,the vertices of G can be colored with k colors.Let m_i be the number of vertices colored with color i for i=1,2,…k.Then $n=m_1+m_2+\dots\dots\dots+m_k$.Since G has the maximum number of edges,very pair of vertices that are colored with different colors is adjacent.G is then the complete k-partite graph $K_{m1,m2}$………mk.

If we denote the number of edges in a complete K-partite graph $K_{m1,m2}$………mk by $A(m_1,m_2\ldots m_k)$,then

$A(m_1,m_2 \ldots m_k)=m_1m_2+ A(m_{1+}m_{2,}\ldots m_k)$

Suppose that in G any two by these numbers differ by more than one,say $m_2+2\le m_{1.}$ Then as before,we consider a new graph $G=Km_{1-1,m2+1,m3\ldots\ldots.mk.}$

The number of edges in G is $A(m_{1}\text{-}1\ m_2{}^{+}1\ ,\ m_3\ldots m_k)= (m_{1}\text{-}1)(\ m_2{}^{+}1)+A(m_1,+m_{2,}\ \ldots m_k)$

The number of edges in G minus the number of edges in G is $(m_{1}\text{-}1)(\ m_2{}^{+}1)\text{-}\ m_1m_2 = m_{1}\text{-}m_{2}\text{-}1$

Since,m1≥m2+2,we have $m_{1}\text{-}m_{2}\text{-}1\ge 1$.So G has atleast one more edge than G,hence G could not have the maximum number of edges.The assumption that $m_2+2\le m_{1,}$ leads to a contradiction,thus any two of the numbers m_i and m_j differ by at most one.

VII. CONCLUSION

Obviously this field still offers a huge number of problems. The most obvious of which are finding more Ramsey numbers and improving the bounds we presently know. However, there are many related problems that go beyond this. We will highlight a few of these problems, and the reader should note that, in most cases, there is a cash reward available for the solutions.

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