More on the Diophantine Equation $27^x + 2^y = Z^2$

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Abstract— In this paper, we show that (0, 3, 3) is a unique non-negative integer solution for the Diophantine equation, $27^x+2^y=z^2$, where x, y and z are non-negative integers.

Key words: Catalan Conjectures, Diophantine Equation

I. INTRODUCTION

In 2007, Acu [1] proved that (3, 0, 3) and (2, 1, 3) are only two solutions in non-negative integers of the Diophantine equation $2^x+5^y=z^2$. In 2013, Sroysang [2] proved that more on the Diophantine equation $2^x + 32^y = z^2$ has non-negative integer (3, 0, 3) is a unique non- negative integer solution. In this paper we will show that the Diophantine equation $27^x + 2^y = z^2$ has non-negative integer (0, 3, 3) is a unique non-negative integer solution.

II. PRELIMINARIES

In 1844, Catalan [3] conjectures that the Diophantine equation $a^x - b^y=1$ has a unique integer solution with min{a, b, x, y}>1. The solution (a, b, x, y) is (3, 2, 2, 3). This conjecture was proven by Mihailescu [4] in 2004

A. *Preposition 2.1*

([5]). (3, 2, 2, 3) is a unique solution (a, b, x, y) of the Diophantine equation $a^x - b^y=1$, where a, b, x and y are integers with min{a, b, x, y}>1

B. *Lemma 2.2*

[1] (3, 3) is a unique solution of (y, z) for the Diophantine equation $1+2^y=z^2$. Where y and z are non-negative integers.

C. *Lemma 2.3*

The Diophantine equation $27^x+1=z^2$ has no non-negative integer solution where x and z are non-negative integers.

1) *Proof*

Suppose that there are non-negative integers x and z such that $27^x+1=z^2$. If x=0, then $z^2=2$ which is impossible. Then $x \geq 1$. Thus, $z^2=27^x+1\geq 27^1+1=28$, then z>5. Now we consider on the equation $z^2-27^x=1$. By preposition 2.1, we have x=1. Then $z^2=28$. This is a contradiction. Hence, the equation $27^x+1=z^2$ has no non negative integer solution.

III. RESULTS

A. *Theorem 3.1*

(0, 3, 3) is a unique solution (x, y, z) for the Diophantine equation $27^x+2^y=z^2$ where x, y and z non-negative integers.

1) *Proof*

Let x, y and z be non-negative integers such that $27^x+2^y=z^2$. By lemma 2.3, we have y≥1. Thus z is odd then there is a non-negative integer t such that z=2t+1. We obtain that $27^x+2^y=4(t^2+t)+1$. Then $27^x\equiv 1 \pmod 4$. Thus x is even. Then there is a non-negative integer k such that x=2k. We divide the number x into two cases.

- Case x=0. By lemma 2.2, we have y=3 and z=3.
- Case x≥2. Then k≥1. Then $z^2-27^{2k}=2^y$. Then $(z-27^k)(z+27^k) = 2^y$. We obtain that $z-27^k=2^u$, where u is a non-negative integer. Then $z+27^k=2^{y-u}$. It follows that $2(27^k)=2^{y-u}-2^u=2^u(2^{y-2u}-1)$. We divide the number u into two subcases.
- Subcase u=0. Then $z-27^k=1$. Then z is even. This is a contradiction.
- Subcase u=1. Then $2^{y-2}-1=27^k$. It follows that $2^{y-2}=27^k+1\geq 27+1=28$. Thus y≥6. More over $2^{y-2}-27^k=1$. By preposition 2.1, we have k=1, then $2^{y-2}=28$. This is impossible.

Therefore, (0,3,3) is a unique solution (x,y,z) for the equation $27^x+2^y=z^2$

B. *Corollary 3.2*

The Diophantine equation $27^x+2^y=w^4$ has no non-negative integer solution. Where x,y and w are non-negative integers.

1) *Proof*

Suppose that there are non-negative integers x,y and w such that $27^x+2^y=w^4$. Let $z=w^2$. Then $27^x+2^y=z^2$. By lemma 3.1, we have (x,y,z)=(0,3,3). Then $w=z^2=3$. This is a contradiction.

C. *Corollary 3.3*

(0, 1, 3) is a unique solution of (x, y, z) for the Diophantine equation $27^x+8^u=z^2$, where x,u and z are non-negative integers.

1) *Proof*

Let x,y and z are non-negative integers such that $27^x+8^u=z^2$. Let y=3u. Then $27^x+2^y=z^2$. By theorem 3.1 we have (x, y, z)=(0,3,3). Then y=3u=3. Thus u=1. Therefore, (0, 3, 3) is a unique solution (x, u, z) for the equation $27x+8^u=z^2$.

D. *Corollary 3.4*

The Diophantine equation $27^x+32^y=z^2$ has no non-negative integer solution. Where x,u and z are non-negative integers.

1) *Proof*

Suppose that there are non-negative integers x,u and z such that $27^x+32^y=z^2$. Let y=5u. Then $27^x+2^y=z^2$. By theorem 3.1, we have y=5u=3. This is contradiction.

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