Common Fixed Point Theorem for a Pair of Compatible Self Maps of a D*-Metric Space

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Abstract— The purpose of this paper is to establish a common fixed point theorem for a pair of compatible self-maps of a D*-metric space using the contractive modulus and deduce corresponding results for metric spaces.

Key words: D*-Metric Space, Contractive Modulus

I. INTRODUCTION

Generally fixed point theorems were established for selfmaps of metric spaces. Certain fixed point theorems were proved for selfmaps of metrizable topological spaces also since such spaces, for all practical purposes, can be considered as metric spaces.

Recently in 1992 B. C. Dhage [1] has initiated a study of general metric spaces called *D*-metric spaces. Later several researchers have made a significant contribution to the fixed point theorems of D-metric spaces in [2], [3], [4], [5] and [6].

II. PRELIMINARIES

A. Definition 2.1

A function $\phi:[0,\infty)\to[0,\infty)$ as said to be a contractive modulus, if $\phi(0)=0$ and $\phi(t)<t$ for $t>0$.

B. Definition 2.2

Let X be a non-empty set. A function $D^*: X^3\to[0,\infty)$ is said to be a generalized metric or D*-metric on X, if it satisfies the following conditions:

- $D^*(x,y,z)\ge 0$ for all $x,y,z\in X$
- $D^*(x,y,z)=0$ if and only if $x=y=z$
- $D^*(x,y,z)=D^*(\sigma(x,y,z))$ for all $x,y,z\in X$, Where $\sigma(x,y,z)$ is a permutation of the set {x,y,z}
- $D^*(x,y,z)\le D^*(x,y,w)+D^*(w,z,z)$ For all $x,y,z,w\in X$.

The pair (X, D*), where D* is a generalized metric on X is called a D*-metric space or a generalized metric space.

III. MAIN RESULTS

A. Theorem

- Suppose f is a continuous selfmap of a D*-metric space (X, D*). Then f has a fixed point in X if and only if there is a contractive modulus ϕ and a selfmap g of X such that
- f and g are compatible
- $D^*(gx,gy,gy)\le\phi(D^*(fx,fy,fy))$ for all $x,y\in X$
- There is a point $x_0\in X$ and an associate sequence $\{x_n\}$ of x^0 relative to the selfmaps f and g such that the sequence $\{fx_n\}$ converges to some point t of X.
- Further *gt* is the unique common fixed point of *f* and *g*.

B. Proof

To prove the necessary part, suppose that *f* has a fixed point, say, $a\in X$, then $fa=a$. Define $g:X\to X$ by $gx=a$ for all $x\in X$. Now for any $x\in X$, we have

$(gf)x=gfx=a$ and $(fg)x=fgx=fa=a$ for any $x\in X$

Giving that *fg=gf*. So that *f* and *g* are compatible mappings. Now let ϕ be a contractive modulus, then $\phi(0)=0$ and $\phi(t)<t$ for $t>0$ and for any $x,y\in X$, $D^*(gx,gy,gy)=D^*(a,a,a)=0\le\phi(D^*(fx,fy,fy))$

Further an associated sequence of $x_0=a$ relative to the selfmaps *f* and *g* is given by $x_n=a$ for $n=0,1,2,\ldots$ and since the sequence $\{fx_n\}$ is a constant sequence converging to *a*, which is a point in *X*. Thus the conditions (3.2) to (3.4) of the theorem are satisfied.

Conversely, suppose that there is a contractive modulus ϕ and a selfmap g of X satisfying the conditions (3.2) to (3.4) hold. From (3.4) there is an associated sequence $\{x_n\}$ of x_0 relative to the selfmaps f and g such that $fx_n=gx_{n-1}$ for n = 1,2,3… and $fx_n\to t$ as $n\to\infty$ for some $t\in X$. Then since $gx_n=fx_{n+1}$, it follows that $gx_n\to t$ as $n\to\infty$.

Now we shall show that *g* is continuous on *X*. To see this, suppose that $\{y_n\}$ is a sequence in *X* with $y_n\to y$ as $n\to\infty$, $y\in X$. Since *f* is continuous, we have $fy_n\to fy$ as $n\to\infty$. This together with the inequality (3.3), We get $D^*(gy_n,gy,gy)\le\phi(D^*(fy_n,fy,fy))\to 0$ as $n\to\infty$, which implies that $gy_n\to gy$ as $n\to\infty$, showing that g is continuous.

Using the continuity of *f* and *g*, we get $gfx_n\to gt$, $fgx_n\to ft$ as $n\to\infty$. Since $fx_n\to t$, $gx_n\to t$ as $n\to\infty$ and *f* and *g* are compatible, we have $\lim_{n\to\infty}D^*(fgx_n,gfx_n,gfx_n)=0$ which implies that

$$D^*\left(\lim_{n\to\infty}fgx_n,\lim_{n\to\infty}gfx_n,\lim_{n\to\infty}gfx_n\right)=0,$$

It follows that $D^*(ft,gt,gt)=0$, which implies $ft=gt$.

To show that *fg t* = *gf t*, take $z_n=t$ for $n=1,2,3,\ldots$, so that $fz_n\to ft$ and $gz_n\to gt$ as $n\to\infty$. Since *f t=gt* and *f* and *g* are compatible, we get $\lim_{n\to\infty}D^*(fgz_n,gfz_n,gfz_n)=0$ this implies

$$D^*\left(\lim_{n\to\infty}fgz_n,\lim_{n\to\infty}gfz_n,\lim_{n\to\infty}gfz_n\right)=0.$$

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Using the continuity of f and g, we get $gfz_n \to gft$ and $fgz_n \to fgt$ as $n \to \infty$. It follows that $D^*(fgt, gft, gft) = 0$ and hence $fgt = gft$. Consequently, $fft = fgt = gft = ggt$

If possible suppose that $gt \neq ggt$, Then $D^*(gt, ggt, ggt) > 0$ and hence (3.6)

$\phi(D^*(gt, ggt, ggt)) < D^*(gt, ggt, ggt)$ But from (3.3) and (3.5), we get

$$D^*(gt, ggt, ggt) \leq \phi(D^*(ft, fgt, fgt)) = \phi(D^*(gt, ggt, ggt)),$$

contradicting (3.6). Hence gt = ggt. Using this in (3.5), we get $ggt = gt = fgt$, showing that gt is a common fixed point of f and g.

To see that f and g have unique common fixed point, suppose that $u = fu = gu$ and $v = fv = gv$ for some $u, v \in X$. If possible suppose that $u \neq v$. Then $D^*(u, v, v) \neq 0$, so that

– $\phi(D^*(u, v, v)) < D^*(u, v, v)$

From (3.3), we have

$$D^*(u, v, v) = D^*(gu, gv, gv) \leq \phi(D^*(fu, fv, fv)) = \phi(D^*(u, v, v)),$$

contradicting (3.7). Hence $u = v$, proving the theorem.

1) Corollary

Suppose f is a continuous selfmap of a D*-metric space (X, D*). Then f has a fixed point in X if and only if there is a contractive modulus ϕ and a selfmap g of X such that

– fg = gf
– $D^*(gx, gy, gy) \leq \phi(D^*(fx, fy, fy))$ for all $x, y \in X$
– and
– There is a point $x_0 \in X$ and an associated sequence $\{x_n\}$ of x_0 relative to the selfmaps f and g such that the sequence $\{fx_n\}$ converges to some point t of X.

Further gt is the unique common fixed point of f and g.

2) Proof

From the fact that the commutatively implies the compatibility of a pair of self-maps, Corollary follows from Theorem 3.1.

C. Some Consequences of Theorem 3.1

1) Corollary

Suppose f and g are selfmaps of a D*-metric space (X, D*), f is continuous and that there is a contractive modulus ϕ and a positive integer k such that

2) $fg = gf$

3) $D^*(g^k x, g^k y, g^k y) \leq \phi(D^*(fx, fy, fy))$ *for all* $x, y \in X$.

Assume that

4) There is a point $x_0 \in X$ *and an associated sequence* $\{x_n\}$ *of* X_0 *relative to the selfmaps f and* g^k *such that the sequence* $\{fx_n\}$ *converges to some point t of X .*

Then gt is the unique common fixed point of f and g.

5) Proof

From (3.1.2), we get $fg^k = g^k f$ .Thus f and g^k are commuting and hence satisfying the hypothesis of Corollary 3.7, and therefore f and g^k have unique common fixed point, say, b, then $g^k b = b = fb$.

Now

$$g^k gb = g^{k+1} b = gg^k b = gb \text{ and } fgb = gfb = gb$$

This shows that gb is a common fixed point of f and g^k The uniqueness of b implies that gb=b Since fb=b, b is a common fixed point of f and g.

To prove that f and g have unique common fixed point, suppose that u = fu = gu and v = fv = gv for some $u, v \in X$, so that $g^k u = u$ and $g^k v = v$; this shows that u, v are common fixed points of f and g^k. The uniqueness of common fixed points of f and g^k implies u = v.

6) Corollary

Let p be a positive integer. If g is a continuous selfmap of a D*-metric space (X, D*) such that

– $\phi(D^*(g^p x, g^p y, g^p y)) \geq D^*(x, y, y)$ for all $x, y \in X$ and
– there is a point $x_0 \in X$ and an associated sequence $\{x_n\}$ of x_0 relative to the selfmaps g^p and I (where I is the identity map on X) such that the sequence $\{g^p x_n\}$ converges to some point t of X, then g has a unique fixed point in X.

7) Proof

We know that $g^p I = I g^p$. From (6.3.6), we have $D^*(x, y, y) = D^*(Ix, Iy, Iy) \leq \alpha . D^*(g^p x, g^p y, g^p y)$ for all $x, y \in X$. Since g is continuous, g^p is continuous.

Applying Corollary 3.1.1 to the functions g^p and I, we have g and I have unique common fixed point, showing that g has unique fixed point as every point of X is a fixed point of I.

8) Theorem

Suppose f is a continuous selfmap of a D*-metric space (X, D*). Then f has a fixed point in X if and only if there is a contractive modulus ϕ and a continuous selfmap g of X such that

9) f and g are compatible

10) $D^*(gx, gy, gy) \leq \phi(M(x, y))$ *for all* $x, y \in X$

Where

$$M(x, y) = M_{f,g}(x, y) = \max\{ D^*(fx, fy, fy), D^*(fx, gy, gy), D^*(fy, gx, gx) \}$$

And

– There is a point $x_0 \in X$ and an associated sequence $\{x_n\}$ of x_0 relative to the selfmaps f and g such that the sequence $\{fx_n\}$ converges to some point t of X.
– Further gt is the unique common fixed point of f and g.

11) Proof

The necessary part of the theorem is same as in the Theorem 3.1.

Conversely, suppose that there is a contractive modulus ϕ and a continuous selfmap g of X satisfying the conditions (3.1.9) to (3.1.11). From (3.1.11), there is an associated sequence $\{x_n\}$ of x_0 such that $fx_n = gx_{n-1}$ for $n = 1, 2, 3, \ldots$ and $fx_n \to \infty$ as $n \to \infty$, it follows that $gx_n = fx_{n+1} \to t$ as $n \to \infty$.

From (3.1.9) and since $fx_n \to t,\ gx_n \to t$ as $n \to \infty$, we have

$$\lim_{n \to \infty} D^*(fgx_n, gfx_n, gfx_n) = 0,$$

Using the continuity of D*,

$$D^*\left(\lim_{n \to \infty} fgx_n, \lim_{n \to \infty} gfx_n, \lim_{n \to \infty} gfx_n\right) = 0$$

and also using the continuity of f and g, we get

$D^*(ft, gt, gt) = 0$, so that $ft = gt$.

To show that $fgt = gft$, take $z_n = t$ for $n = 1, 2, 3, \ldots$, so that $fz_n \to ft$ and $gz_n \to gt$ as $n \to \infty$. Since $ft = gt$ and *f* and *g* are compatible, we get $\lim_{n \to \infty} D^*(fgz_n, gfz_n, gfz_n) = 0$ which implies that $D^*\left(\lim_{n \to \infty} fgz_n, \lim_{n \to \infty} gfz_n, \lim_{n \to \infty} gfz_n\right) = 0$, using the continuity of *f* and *g*, we get $gfz_n \to gft$ and $fgz_n \to fgt$ as $n \to \infty$, it follows that $D^*(fgt, gft, gft) = 0$, so that $fgt = gft$. Consequently

12) $fft = fgt = gft = ggt$

If possible suppose that $gt \neq ggt$, then $D^*(gt, ggt, ggt) > 0$, which implies

13) $\phi(D^*(gt, ggt, ggt)) < D^*(gt, ggt, ggt)$

But from (3.1.10) and (3.1.12), we get

$$D^*(gt, ggt, ggt) \leq \phi(M(t, gt)),$$

Where

$$M(t, gt) = \max\{D^*(ft, fgt, fgt), D^*(ft, ggt, ggt), D^*(fgt, gt, gt)\}$$

$$= \max\{D^*(gt, ggt, ggt), D^*(gt, ggt, ggt), D^*(ggt, gt, gt)\}$$

$$= D^*(gt, ggt, ggt)$$

That is,

$D^*(gt, ggt, ggt) \leq \phi(D^*(gt, ggt, ggt))$, contradiction to (3.1.13). Hence $gt = ggt$, showing that *gt* is a common fixed point of *f* and *g*.

To prove that *f* and *g* have unique common fixed point, suppose that $u = fu = gu$ and $v = fv = gv$ for some $u, v \in X$. If $u \neq v$, then $D^*(u, v, v) > 0$, which implies that

14) $\phi(D^*(u, v, v)) < D^*(u, v, v)$

From (3.1.10), we get

$$D^*(u, v, v) = D^*(gu, gv, gv) \leq \phi(M(u, v))$$

$$= \phi[\max\{D^*(fu, fv, fv), D^*(fu, gv, gv), D^*(fv, gu, gu)\}]$$

$$= \phi[\max\{D^*(u, v, v), D^*(u, v, v), D^*(v, u, u)\}]$$

$$= \phi(D^*(u, v, v))$$

$D^*(u, v, v) \leq \phi(D^*(u, v, v))$, contradicting (3.1.14). Hence u = v. This completes the proof.

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