On Ternary Quadratic Diophantine Equation $4x^2 + 4y^2 - 7xy = 96z^2$

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Abstract— Five different methods of the non-zero integral solution of the ternary quadratic homogeneous Diophantine equation $4x^2+4y^2-7xy = 96z^2$ are obtained. Some interesting relations among the special numbers and the solutions are exposed.

Key words: Ternary Quadratic, homogenous cone, integer solution, special numbers.
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NOTATIONS USED

- $t_{m,n}$= Polygonal number of rank n with sides m.
- $Ct_{m,n}$ = Centered Polygonal number of rank n with sides m
- s_n = Star number
- p_n = Pronic number
- G_n= Gnomonic number

I. INTRODUCTION

The number theory is queen of Mathematics. In particular, the Diophantine equations have a blend of attracted interesting problems. For an extensive review of variety of problems, one may refer to [3-12]. In this work, we are observed another interesting five different methods of the non-zero integral solutions the ternary quadratic homogenous Diophantine equations $4x^2+4y^2-7xy = 96z^2$. Further, some elegant properties among the special numbers and the solutions are observed.

II. METHOD OF ANALYSIS

The ternary quadratic homogenous Diophantine equations to be solve is

$$4x^2+4y^2-7xy = 96z^2 \quad (1)$$

The substitution of linear transformations

$$x = u + v \text{ and } y = u - v \ (u \neq v \neq 0) \quad (2)$$

in (1) leads to

$$u^2 + 15v^2 = 96z^2 \quad (3)$$

The above equation is solved through different methods and different patterns of integer solution to (1) are obtained.

A. Pattern: I

Write 96 as

$$96 = (9 + i\sqrt{15})(9 - i\sqrt{15}) \quad (4)$$

Assume $z = a^2 + 15b^2$, where a, b >0 (5)

Using (4) and (5) in (3) and applying the method of factorization, define

$$(u + i\sqrt{15}v) = (9 + i\sqrt{15})(a + i\sqrt{15}b)^2 \quad (6)$$

Equating the real and imaginary parts, we have

$$u = u(a, b) = 9a^2 - 135b^2 - 30ab$$
$$v = v(a, b) = a^2 - 15b^2 + 18ab$$

Substituting the above u and v in equation (2) the value of x and y are given by

$$x = x(a, b) = 10a^2 - 150b^2 - 12ab$$
$$y = y(a, b) = 8a^2 - 120b^2 - 48ab \quad (7)$$

Thus (5) and (7) represents non-zero distinct integral solutions of (1) in two parameters.

1) Properties:

1) $x(a, 1) - t_{22,a} + t_{10,a} + 4t_{4,a} \equiv 0 \pmod 3$
2) $z(a, a+1) - 10t_{4,a} - S_a - 18G_a \equiv 0 \pmod 5$
3) $y(1, b) + 120t_{4,b} - 24G_b \equiv 0 \pmod 9$
4) $x(a, 1) + z(a, 1) - 11t_{4,a} - Sa + 6Pa \equiv 0 \pmod 2$
5) $y(2, b) + S_b + t_{230,b} - t_{162,b} + 80t_{4,b} \equiv 0 \pmod{31}$.
6) $x(1, b) + 150t_{4,b} + G_{6,b} \equiv 0 \pmod{11}$
7) $z(b, b+1) - 16t_{4,b} - G_{15,b} \equiv 0 \pmod 2$
8) $y(a, 3) - 8t_{4,a} + G_{72,a} + 1079 = 0$.

B. Pattern: II

Consider (3) as $u^2 - 81z^2 = 15(z^2 - v^2)$ (8)

Write (8) in the form of ratio as

$$\frac{u+9z}{z-v} = \frac{15(z+v)}{u-9z} = \frac{\alpha}{\beta}, \beta \neq 0$$

This is equivalent to the following two equations

$$-\alpha u + 15\beta v + z(15\beta + 9\alpha) = 0$$
$$\beta u + \alpha v + z(9\beta - \alpha) = 0$$

On employing the method of cross multiplication, we get.

$$u = -9\alpha^2 + 135\beta^2 - 30\alpha\beta$$
$$v = -\alpha^2 + 15\beta^2 + 18\alpha\beta \quad (9)$$
$$z = -\alpha^2 - 15\beta^2 \quad (10)$$

Substituting the values of u and v from (9) in (2) the non-zero distinct integer values of x, y are given by

$$x = x(\alpha,\beta) = -10\alpha^2 + 150\beta^2 - 12\alpha\beta$$
$$y = y(\alpha,\beta) = -8\alpha^2 + 120\beta^2 - 48\alpha\beta \quad (11)$$

Thus (10) and (11) represent the non- zero distinct integer solution of equation (1) in two parameters.

1) Properties:

1) $z(\beta + 3, \beta + 3) + S_\beta + 10t_{4,\beta} + G_{45\beta} \equiv 0 \pmod 2$
2) $y(3, 2\beta) - C\,t_{16,\beta} - 472t_{4,\beta} + G_{148\beta} \equiv 0 \pmod 2$
3) $x(\alpha, 1) + 10t_{4,\alpha} + G_{6\alpha} \equiv 0 \pmod{151}$
4) $x(2\alpha, 3) + S_\alpha + 34t_{4,\alpha} + G_{39\alpha} \equiv 0 \pmod{1349}$
5) $z(\beta - 2, \beta - 2) + S_\beta + 10t_{4,\beta} + G_{29\beta} \equiv 0 \pmod 5$
6) $x(2, \beta) - 150t_{4,\beta} + G_{12,\beta} + 39 = 0$
7) $y(4, \beta) - 120t4, \beta + G96, \beta + 127 = 0$
8) $z(\alpha+1, \alpha+1) + 16t_{4,\alpha} + G_{16,\alpha} \equiv 0 \pmod 3$

Note:

9) Also be expressed in the form of ratio in three different wags as follows.

$$\frac{(u+9z)}{15(z+v)} = \frac{z-v}{u-9z} = \frac{\alpha}{\beta}, \beta \neq 0.$$

$$\frac{(u+9z)}{z+v} = \frac{15(z-v)}{u-9z} = \frac{\alpha}{\beta}, \beta \neq 0.$$

$$\frac{u+9z}{15(z-v)} = \frac{z+v}{u-9z} = \frac{\alpha}{\beta}, \beta \neq 0.$$

Repeating the analysis as above, we get different pattern of solution to (1).

C. Pattern: III

Rewrite (3) as $15v^2 = 96z^2 - u^2$ (12)

Write 15 as,

$15 = (2\sqrt{10}+5)(2\sqrt{10}-5)$ (13)

Assume $v = 96a^2 - b^2$ (14)

Using (13) and (14) in (12) and employing the method of factorization, we write

$(2\sqrt{10}\,z + u) = (2\sqrt{10} + 5)(2\sqrt{10}\,a + b)^2$

Equating the rational and irrational parts, we have

$z = z(a, b) = 40a^2 + b^2 + 10ab$ (15)

$u = u(a, b) = 200a^2 + 5b^2 + 80ab$ (16)

Substituting (14) & (16) in (2), the values of x and y are

$x = x(a, b) = 296a^2 + 4b^2 + 80ab$

$y = y(a, b) = 104a^2 + 6b^2 + 80ab$ (17)

Thus (17) and (15) represent the integer solution to (1)

1) Properties:

1) $x(a, 1) - 296t_{4,a} - G_{40a} \equiv 0 \pmod 5$
2) $z(3a, 1) - 366\, t_{4,a} - G_{12a} + Sa = 1$
3) $x(a, 2) - z(a, 2) - 256\, t_{4,a} + G_{80a} \equiv 0 \pmod{13}$
4) $y(a,3) - t_{210,\,a} - t_{10,\,a} + 4t_{4,a} - G_{170a} - G_{3a} \equiv 0 \pmod 2$
5) $z(6a, 1) - 1440t_{4,a} - G_{30a} \equiv 0 \pmod 2$
6) $y(4, b) - 6\, t_{4,b} - G_{160,b} \equiv 0 \pmod 5$
7) $z(-a, -a) - 51\, t_{4,a} = 0.$
8) $x(-2, b) - 4t_{4,b} + G_{80,b} \equiv 0 \pmod 3$

III. CONCLUSION

In this paper, we have presented different patterns of integer solutions to the ternary quadratic equation $4x^2+4y^2-7xy = 96z^2$ representing the cone. As thesis Diophantine equations are rich in variety, one may attempt to find integer solutions to other choices of equations along with suitable properties.

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