Common Fixed Point of Mappings using Sequentially Weak Compatible Mapping

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Abstract— In this paper, we prove a common fixed point theorem for a pair of weakly compatible mappings using concept of sequentially weak contraction in which sequence of function is uniformly convergent to a continuous function, which generalise the results of Moradi et. al. [9].

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I. INTRODUCTION

In 1922, the Polish mathematician, Banach proved a common fixed point theorem, which ensures the existence and uniqueness of a fixed point under appropriate conditions. This result of Banach is known as Banach's fixed point theorem or Banach contraction principle, which states that "Let (X, d) be a complete metric space. If T satisfies

(1.1) $d(Tx, Ty) \leq k\, d(x, y)$

for each x, y in X, where $0 < k < 1$, then T has a unique fixed point in X." This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering.

This principle is basic tool in fixed point theory.

A. *Definition 1.1*

Two self-mappings f and g of a metric space (X, d) are said to be weakly commuting if $d(fgx, gfx) \leq d(gx, fx)$ for all x in X. Further, Jungck [7] introduced more generalized commutativity, so called compatibility, which is more general than that of weak commutativity.

B. *Definition 1.2*

Two self-mappings f and g of a metric space (X, d) are said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.

C. *Definition.1.3*

A mapping $T : X \to X$, where (X, d) is a metric space, is said to be sequencially weakly contraction if

(1.2) $d(Tx, Ty) \leq d(x, y) - f_n(d(x,y))$
(f_n:I (interval or subset of R)   R)

where x,y X and $f_n(t)$ is a sequence of function which converges uniformly to t, and monotonic function such that $f_n(t) = 0$ if and only if $t = 0$.

In 2012, Moradi et. al. [9] proved the following Theorem:

D. *Theorem 1.4.*

Let T be a self-mapping on a complete metric space (X, d) satisfying the following:

(1.3) $\psi(d(Tx, Ty)) \leq \psi(N(x, y)) - \varphi(N(x, y))$,

where $N(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+ d(y,Tx)}{2}\}$,

for all x,y∈ X (generalized $(\psi - \varphi)$ weakly contractive), where $\varphi : [0, \infty) \to [0, \infty)$ is a mapping with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ and $\lim_{n\to\infty} t_n = 0$, if $\{t_n\}$ is bounded and $\lim_{n\to\infty} \varphi(t_n) = 0$ and $\psi : [0, \infty) \to [0, \infty)$ is a mapping with $\psi(0) = 0$ and $\psi(t) > 0$ for all $t > 0$.

Also suppose that either

(a) ψ is continuous

or

(b) ψ is monotonic non-decreasing and for all $k > 0$, $\varphi(k) > \psi(k) - \psi(k-)$, where $\psi(k-)$ is the left limit of ψ at k.

Then T has a unique common fixed point.

Now, we prove our results on metric space for pair of sequentially weak compatible mappings.

II. MAIN RESULT

A. *Theorem 2.1*

Let f and g be self-mappings of a metric space (X, d) satisfying the following:

1) $gX \subset fX$,
2) gX or fX is complete,
3) $\psi(d(gx,gy)) \leq \psi(d(fx, fy)) - f_n(d(fx,fy))$,

(f_n : I (interval or subset of R) $\to$ R) for all x, y ∈ X

where : $[0, \infty) \to [0, \infty)$ is mappings with $\psi(0) = 0$, $f_n(t) > 0$ also, $f_n(t)$ is a uniformly convergent sequence which converges to ψ (t) and $\psi(t) > 0$ for all $t > 0$.

Suppose also that either

ψ is continuous and $\lim_{n\to\infty} t_n = 0$, if $\lim_{n\to\infty} f_n(t_n) = 0$.

or

ψ is monotonic non-decreasing and $\lim_{n\to\infty} t_n = 0$, if $\{t_n\}$ is bounded and $\lim_{n\to\infty} f_n(t_n) = 0$.

Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$. From (2.1), one can construct sequences $\{x_n\}$ and $\{y_n\}$ in X by

$y_n = fx_{n+1} = gx_n$, n = 0, 1, 2, ...

Moreover, we assume that if $y_n = y_{n+1}$ for some $n \in \mathbb{N}$, then there is nothing to prove. Now, we assume that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$.

Substituting $x = x_{n+1}$ and $y = x_n$ in (2.3), we have

(2.4) $\psi(d(y_{n+1}, y_n)) = \psi(d(gx_{n+1},gx_n))$
$\leq \psi(d(fx_{n+1}, fx_n)) - f_n(d(fx_{n+1},fx_n))$
$= \psi(d(y_n, y_{n-1})) - f_n(d(y_n,y_{n-1}))$

for all $n \in \mathbb{N}$ and hence the sequence $\{ \psi(d(y_{n+1}, y_n)) \}$ is monotonic decreasing and bounded below. Thus, there exists $r \geq 0$ such that $\lim_{n\to\infty} \psi(d(y_{n+1}, y_n)) = r$.

From (2.4), we deduce that

(2.5) $0 \leq f_n(d(y_n,y_{n-1}))$

$\leq \psi(d(y_n, y_{n-1})) - \psi(d(y_{n+1}, y_n))$.

Letting $n \to \infty$ in the above inequality,

we get $\lim_{n\to\infty} f_n((d(y_n, y_{n-1})) = 0$.

If (a) holds, then by hypothesis $\lim_{n\to\infty} d(y_n, y_{n-1}) = 0$.

If (b) holds, then from (2.5), we have

$d(y_{n+1}, y_n) < d(y_n, y_{n-1})$, for all $n \in \mathbb{N}$.

Hence $\{d(y_{n+1}, y_n)\}$ is monotonically decreasing and bounded below.

By hypothesis, $\lim_{n\to\infty} d(y_n, y_{n-1}) = 0$.

Therefore, in every case, we conclude that

(2.6) $\lim_{n\to\infty} d(y_n, y_{n-1}) = 0$.

Now, we claim that $\{y_n\}$ is a Cauchy sequence. Indeed, if it is false, then there exists $\varepsilon > 0$ and the subsequences $\{y_{m(k)}\}$ and $\{y_{n(k)}\}$ of $\{y_n\}$ such that n(k) is minimal in the sense that $n(k) > m(k) > k$ and $d(y_{m(k)}, y_{n(k)}) \geq \varepsilon$ and by using the triangular inequality, we obtain

$$\varepsilon \leq d(y_{m(k)}, y_{n(k)}) \leq d(y_{m(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{n(k)})$$
$$\leq d(y_{m(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)}) + d(y_{m(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{n(k)})$$

(2.7) $< 2d(y_{m(k)}, y_{m(k)-1}) + \varepsilon + d(y_{n(k)-1}, y_{n(k)})$.

Letting $k \to \infty$ in the above inequality and using (2.6), we get

(2.8) $\lim_{k\to\infty} (d(y_{m(k)}, y_{n(k)}) = \lim_{k\to\infty} (d(y_{m(k)-1}, y_{n(k)-1}) = \varepsilon$.

For all $k \in \mathbb{N}$, from (2.3), we have

(2.9) $(d(y_{m(k)},y_{n(k)})) \leq \psi(d(y_{m(k)-1}, y_{n(k)-1})) - f_n(d(y_{m(k)-1},y_{n(k)-1}))$

If (a) holds, then

$\lim_{k\to\infty} \psi(d(y_{m(k)-1}, y_{n(k)-1})) = \lim_{k\to\infty} \psi(d(y_{m(k)}, y_{n(k)})) = \psi(\varepsilon)$,

Now, from (2.9), we conclude that

$\lim_{k\to\infty} f_n(d(y_{m(k)-1}, y_{n(k)-1}) = 0$.

By hypothesis $\lim_{k\to\infty} d(y_{m(k)-1}, y_{n(k)-1}) = 0$, a contradiction. (Using (2.8))

If (b) holds, then from (2.9), we have

$\varepsilon < d(y_{m(k)}, y_{n(k)}) < d(y_{m(k)-1}, y_{n(k)-1})$, and so

$d(y_{m(k)}, y_{n(k)}) \to \varepsilon^+$ and $d(y_{m(k)-1}, y_{n(k)-1}) \to \varepsilon^+$ as $k \to \infty$.

Hence $\lim_{k\to\infty} \psi(d(y_{m(k)-1}, y_{n(k)-1})) = \lim_{k\to\infty} \psi(d(y_{m(k)}, y_{n(k)})) = \psi(\varepsilon^+)$, where $\psi(\varepsilon^+)$ is the right limit of ψ at ε.

Therefore, from (2.9), we get $\lim_{k\to\infty} f_n(d(y_{m(k)-1}, y_{n(k)-1})) = 0$.

By hypothesis $\lim_{k\to\infty} d(y_{m(k)-1}, y_{n(k)-1}) = 0$, a contradiction.

Thus $\{y_n\}$ is a Cauchy sequence.

Since fX is complete, so there exists a point $z \in fX$ such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_{n+1} = z$.

Now, we show that z is the common fixed point of f and g. Since $z \in fX$, so there exists a point $p \in X$ such that $fp = z$.

If (a) holds, then from (2.3), for all $n \in \mathbb{N}$, we have

$\psi(d(fp, gp)) = \lim_{n\to\infty} \psi(d(gp,gx_n)) \leq \lim_{n\to\infty} \psi(d(fp, fx_n)) - \lim_{k\to\infty} f_n (d(fp,fx_n))$

$\leq \lim_{n\to\infty} \psi(d(fp, fx_n))$.

(2.10) $\psi(d(fp, gp)) \leq \lim_{n\to\infty} \psi(d(fp, fx_n))$.

Using condition (a) and $\lim_{n\to\infty} y_n = z$, we get

$\psi(d(fp, gp)) \leq \psi(d(z, z)) = \psi(0) = 0$ and so $d(gp, fp) = 0$ (note that f_n and ψ are non-negative with $f_n(0)= \psi(0) = 0$), which implies that $gp = fp = z$.

If (b) holds, then from (2.7), we have

$\psi(d(fp, gp)) = \lim_{n\to\infty} \psi(d(gp,gx_n)) \leq \lim_{n\to\infty} \psi(d(fp, fx_n)) - \lim_{k\to\infty} f_n(d(fp,fx_n))$

(2.11) $\psi(d(fp, gp)) = 0$ (since f_n converges uniformly to ψ)

$d(fp, gp) = 0$, which implies that $fp = gp = z$(say).

Now, we show that $z = fp = gp$ is a common fixed point of f and g. Since $fp = gp$ and f, g are weakly compatible maps, we have $fz = fgp = gfp = gz$.

We claim that $fz = gz = z$.

Let, if possible, $gz \neq z$.

If (a) holds, then from (2.3), we have

$$\psi(d(gz, z)) = \psi(d(gz, gp)) \leq \psi(d(fz, fp)) - f_n(d(fz,fp)) = \psi(d(gz, z)) - f_n(d(gz,z)) < \psi(d(gz, z)),$$ a contradiction.

If (b) holds, then we have

$d(gz, z) < d(gz, z)$, a contradiction.

Hence $gz = z = fz$, so z is the common fixed point of f and g.

For the uniqueness, let u be another common fixed point of f and g, so that $fu = gu = u$.

We claim that $z = u$.

Let, if possible, $z \neq u$.

If (a) holds and $n \to \infty$ then from (2.3), we have

$$\psi(d(z, u)) = \psi(d(gz, gu)) \leq \psi(d(fz, fu)) - f_n(d(fz,fu)) = \psi(d(z, u)) - f_n(d(z,u)) < \psi(d(z, u)),$$ a contradiction.

If (b) holds, then we have

$d(z, u) < d(z, u)$, a contradiction.

Thus,$(d(z, u)=0$ i.e we get $z = u$.

Hence z is the unique common fixed point of f and g.

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