Solving Fuzzy Matrix Games Defuzzificated by Trapezoidal Parabolic Fuzzy Numbers

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Abstract— The matrix game theory gives a mathematical background for dealing with competitive or antagonistic situations arise in many parts of real life. Matrix games have been extensively studied and successfully applied to many fields such as economics, business, management and e-commerce as well as advertising. This paper deals with two-person matrix games whose elements of pay-off matrix are fuzzy numbers. Then the corresponding matrix game has been converted into crisp game using defuzzification techniques. The value of the matrix game for each player is obtained by solving corresponding crisp game problems using the existing method. Finally, to illustrate the proposed methodology, a practical and realistic numerical example has been applied for different defuzzification methods and the obtained results have been compared

.***Key words:*** Fuzzy number, trapezoidal fuzzy number, probabilic fuzzy number, Fuzzy payoff, Defuzzification , game theory , two-person matrix games

I. INTRODUCTION

In many real world practical problems with competitive situation, it is required to take the decision where there are two or more opposite parties with conflicting interests and the action of one depends upon the action which is taken by the opponent. A great variety of competitive situation is commonly seen in everyday life viz., in military battles, political campaign, elections economics, business, management and e-commerce, etc. Game theory is a mathematical way out for finding of conflicting interests with competitive situations, which includes players or decision makers who select different strategies from the set of admissible strategies.

In this paper, we have treated imprecise parameters considering fuzzy sets/fuzzy numbers. To handle the problem with such types of imprecise parameters, generally stochastic, fuzzy and fuzzy-stochastic approaches are applied and the corresponding problems are converted into deterministic problems for solving them. In the last few years, several attempts have been made in the existing literature for solving game problem with fuzzy payoff. . Fuzziness in game problem has been well discussed by Campos [1]. Sakawa and Nishizaki [2] introduced max-min solution procedure for multi-objective fuzzy games. Based on fuzzy duality theory [3, 4, 5], Bector et al. [6, 7], and Vijay et al. [8] proved that a two person zero-sum matrix game with fuzzy goals and fuzzy payoffs is equivalent to a pair of linear programming problems. [9] Vahidi, J. and Rezvani, S.: 2013, Arithmetic operations on trapezoidal fuzzy numbers, Jounal Nonlinear Analysis and Applications 2013, Article ID jnaa–00111, In this paper, two person matrix games have taken into consideration. The element of payoff matrix is considered to be fuzzy number [10]. Then the corresponding problem has been converted into crisp equivalent two person matrix game using different defuzzification methods [11]. . Finally, to illustrate the methodology, a numerical example has been applied for different defuzzification methods and the computed results have been compared.

II. DEFINITION AND PRELIMINARIES

A. Definition 2.1:

Let be a non-empty set. A fuzzy set $\tilde{A}$ is defined as the set of pairs, $\tilde{A} = \{x, \mu_{\tilde{A}}(x) : x \in X\}$ where $\mu_{\tilde{A}}(x) : [0,1]$ is a mapping and $\mu_{\tilde{A}}(x)$ is called the membership function of $\tilde{A}$. The value $\mu_{\tilde{A}}(x)$ =0 is used to represent for complete non-membership, whereas $\mu_{\tilde{A}}(x)$ =1 is used to represent for complete membership. The values in between zero and one are used to represent intermediate degrees of membership.

B. Definition 2. 2:

A fuzzy set $\tilde{A}$ is called convex if for all $x_1, x_2 \in X$ such that $\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}$,where $\lambda \in [0,1]$.

C. Definition 2.3:

The set of elements that belong to the fuzzy set $\tilde{A}$ at least to the degree α is called the α -level set or α -cut and is given by $\tilde{A}_\alpha = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\}$. If $\tilde{A}_\alpha = \{x \in X : \mu_{\tilde{A}}(x) > \alpha\}$, it is called strong α -level set or strong α -cut.

D. Definition 2.4:

A fuzzy set $\tilde{A}$ is called a normal fuzzy set if there exist at least one $x \in X$ such that $\mu_{\tilde{A}}(x) = 1$.

E. Definition 2.5:

A fuzzy number is a fuzzy set on the real line R , must satisfy the following conditions.

1) There exists at least one $x_0 \in R$ for which $\mu_{\tilde{A}}(x_0) = 1$. .
2) $\mu_{\tilde{A}}(x)$ is pair wise continuous.
3) $\tilde{A}$ must be convex and normal.

F. Definition 2.6:

A trapezoidal fuzzy fuzzy number (TrFN) $\tilde{A}$ is a normal fuzzy number represented by the quadruplet (a,b,c,d) where $a \le b \le c \le d$ are real numbers and its membership function $\mu_{\tilde{A}}(x): X \to [0,1]$ is given below

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a}{b-a}, & if \ \ a \le x \le b \\ 1, & if \ \ b \le x \le c \\ \frac{x-d}{c-d}, & if \ \ c \le x \le d \\ 0, & else \end{cases}$$

G. Definition 2.7:

A trapezoidal parabolic fuzzy number (TrPFN) $\tilde{A}$ is a normal fuzzy number represented by the quadruplet (a,b,c,d) where $a \le b \le c \le d$ are real numbers and its membership function $\mu_{\tilde{A}}(x): X \to [0,1]$ is given below

$$\mu_{\tilde{A}}(x) = \begin{cases} 1-\left(\frac{b-x}{b-a}\right)^2, & if \ \ a \le x \le b \\ 1, & if \ \ b \le x \le c \\ 1-\left(\frac{x-c}{d-c}\right)^2, & if \ \ c \le x \le d \\ 0, & else \end{cases}$$

III. FUZZY GAME MATHEMATICAL MODELING

A. Payoff Matrix:

Let $A_i \in \{A_1, A_2, A_3, A_4, ..., A_m\}$ be a pure strategy available for player A and $B_j \in \{B_1, B_2, B_3, B_4, ..., B_n\}$ be a pure strategy available for player B. When player A chooses a pure strategy A_i and the player B chooses a pure strategy B_j, then g_{ij} is the payoff for player A and $-g_{ij}$ be the payoff for the player B. The two-person zero-sum matrix game G can be represented as a pay-off matrix $G = [g_{ij}]_{m\times n}$.

B. Fuzzy Payoff Matrix:

Let $A_i \in \{A_1, A_2, A_3, A_4, ..., A_m\}$ be a pure strategy available for player A and $B_j \in \{B_1, B_2, B_3, B_4, ..., B_n\}$ be a pure strategy available for player B. Here, it is assumed that each player has his/her choices from amongst the pure strategies. Also, it is assumed that player A is always the gainer and player B is always the loser. That is, all payoffs are assumed in terms of player A. Let $\tilde{g}_{ij}$ be the fuzzy payoff which is the gain of player A from player B if player A chooses strategy A_i where as player B chooses B_j.Then the fuzzy payoff matrix of player A and B is $\tilde{G} = [\tilde{g}_{ij}]_{m\times n}$.

C. Mixed Strategy:

Let us consider the fuzzy matrix game whose payoff matrix is $\tilde{G} = [\tilde{g}_{ij}]_{m\times n}$. The mixed strategy for the player-A, is denoted by $\xi = (x_1, x_2, x_3, ... x_m)$, where $x_i \ge 0, i = 1,2,3,....m$, and $\sum x_i = 1$. It is to be noted that $e_i^m = (0,...0,1,0,...0)', \ x_i \ge 0, i = 1,2,3,....m$ represent the pure strategy for the player-A and $\xi = \sum_{i-1}^{m} e_i^m x_i$. If $S_m = \left\{\xi : x_i \ge 0, \sum_{i=1}^{m} x_i = 1\right\}$ then $S_m \in E_m$.

Similarly, The mixed strategy for the player-B, is denoted by $\eta = (y_1, y_2, y_3, ... y_n)$, where $y_j \ge 0, \ j = 1,2,3,....n$, and $\sum y_j = 1$. It is to be noted that $e_j^n = (0,...0,1,0,...0)', \ y_j \ge 0, \ j = 1,2,3,....n$ represent the pure strategy for the player-B and $\eta = \sum_{j-1}^{n} e_j^n y_j$. If $S_n = \left\{\eta : y_j \ge 0, \sum_{j=1}^{n} y_j = 1\right\}$ then $S_n \in E_n$. Where S_m and S_n are the spaces of mixed strategies for the player-A and player-B respectively.

D. Maximin-Minimax Principle or Maximin-Minimax Criteria of Optimality for Fuzzy Payoff Matrix:

Let the player A's payoff matrix be $[g_{ij}]_{m\times n}$. If player A takes the strategy A_i then surely he/she will get at least $i = 1,2,3,....m$ for taking any strategy by the opponent player B . Thus by the maximin-minimax criteria of optimality, the player A will choose that strategy which corresponds to the best of these worst outcomes $\min_j DFV(\tilde{g}_{1j}), \min_j DFV(\tilde{g}_{2j}),..... \min_j DFV(\tilde{g}_{mj})$ .Thus the maximin value for player A is given by $\max_i\left(\min_j DFV(\tilde{g}_{ij})\right)$.Similarly, player B will choose that strategy which corresponds to the best (minimum) of the worst outcomes (maximum losses) $\max_i DFV(\tilde{g}_{i1}), \max_i DFV(\tilde{g}_{i2}),..... \max_i DFV(\tilde{g}_{in})$. Thus the maximin value for player B is given by $\min_j\left(\max_i DFV(\tilde{g}_{ij})\right)$.Here , $DFV(\tilde{g}_{ij})$ represents defuzzified value of the fuzzy number $\tilde{g}_{ij}$

1) Theorem 1:

If a matrix game possesses a saddle point, it is necessary and sufficient that

$$\max_i\left(\min_j DFV(\tilde{g}_{ij})\right) = \min_j\left(\max_i DFV(\tilde{g}_{ij})\right).$$

a) Definition 3.1:

A pair (ξ, η) of mixed strategies for the players in a matrix game is called a situation in mixed strategies. In a situation (ξ, η) of mixed strategies each usual situation (i, j) in pure strategies becomes a random event occurring with probabilities x_i, y_j. Since in the situation (i, j), player-A receives a payoff $DFV(\tilde{g}_{ij})$, the mathematical expectation of his payoff under (ξ, η) is equal to

$$E(\xi,\eta) = \sum_{i=1}^{m}\sum_{j=1}^{n} DFV(g_{ij}) x_i y_j .$$

2) *Theorem 2;*

Let $E(\xi,\eta)$ be such that both $\min_{\eta\in S_n}\max_{\xi\in S_m} E(\xi,\eta)$ and $\max_{\xi\in S_m}\min_{\eta\in S_n} E(\xi,\eta)$ exist, then

$$\min_{\eta\in S_n}\max_{\xi\in S_m} E(\xi,\eta) \geq \max_{\xi\in S_m}\min_{\eta\in S_n} E(\xi,\eta).$$

E. Saddle Point Of A Function:

Let $E(\xi,\eta)$ be a function of two variables (vectors) ξ and η in S_m and S_n respectively. The point $(\xi_0,\eta_0), \xi_0 \in S_m, \eta_0 \in S_n$ is said to be the saddle point of the function $E(\xi,\eta)$ if

$$E(\xi,\eta_0) \leq E(\xi_0,\eta_0) \leq E(\xi_0,\eta).$$

1) *Theorem 3:*

Let $E(\xi,\eta)$ be a function of two variables $\xi \in S_m$ and $\eta \in S_n$ such that $\max_{\xi\in S_m}\min_{\eta\in S_n} E(\xi,\eta)$ and $\min_{\eta\in S_n}\max_{\xi\in S_m} E(\xi,\eta)$ exist. Then the necessary and sufficient condition for the existence of a saddle point (ξ_0,η_0) of $E(\xi,\eta)$ is that

$$E(\xi_0,\eta_0) = \max_{\xi\in S_m}\min_{\eta\in S_n} E(\xi,\eta) = \min_{\eta\in S_n}\max_{\xi\in S_m} E(\xi,\eta).$$

F. Value Of A Matrix Game:

The common value of $\max_{\xi\in S_m}\min_{\eta\in S_n} E(\xi,\eta)$ and $\min_{\eta\in S_n}\max_{\xi\in S_m} E(\xi,\eta)$ is called the value of the matrix game with payoff matrix $\tilde{G} = [\tilde{g}_{ij}]_{m\times n}$ and denoted by v(G) or simply v.

1) *Definition 3.2:*

Thus if (ξ,η) is an equilibrium situation in mixed strategies of the game $\langle S_m, S_n, E\rangle$, then ξ^*, η^* are the optimal strategies for the players A and B respectively in the matrix game with fuzzy payoff matrix $\tilde{G} = [\tilde{g}_{ij}]_{m\times n}$. Hence ξ^*, η^* are optimal strategies for the players A and B respectively iff

$$E(\xi,\eta^*) \leq E(\xi^*,\eta^*) \leq E(\xi^*,\eta)\ \forall \xi \in S_m, \eta \in S_n.$$

2) *Definition 3.3:*

– $\min_{\eta} E(\xi,\eta) = E(\xi,\eta^*) \Rightarrow \max_{\xi}\{\min_{\eta} E(\xi,\eta)\} = \max_{\xi} E(\xi,\eta^*) = E(\xi^*,\eta^*)$

– $\max_{\xi} E(\xi,\eta) = E(\xi^*,\eta) \Rightarrow \min_{\eta}\left\{\max_{\xi} E(\xi,\eta)\right\} = \min_{\eta} E(\xi^*,\eta) = E(\xi^*,\eta^*)$

a) Theorem4;

$v = \max_{\xi}\{\min_{j} E(\xi, j)\} = \min_{\eta}\{\max_{i} E(i,\eta)\}$ and the outer extrema are attained at optimal strategies of players.

b) Theorem5:

$$\max_{i}\left(\min_{j} DFV(\tilde{g}_{ij})\right) \leq v \leq \min_{j}\left(\max_{i} DFV(\tilde{g}_{ij})\right).$$

Proof: By the theorem 4, we have $v = \max_{\xi}\{\min_{j} E(\xi, j)\}\forall \xi \in S_m$. But $\max_{\xi}\{\min_{j} E(\xi, j)\} \geq \min_{j} E(\xi, j) \forall \xi \in S_m$. Therefore Letting $\xi = e_i^m$ we have $v \geq \min_{j} E(e_i^m, j) = \min_{j} E(i, j) = \min_{j} DFV(\tilde{g}_{ij})$ and we get $v \geq \min_{j} DFV(\tilde{g}_{ij})$. The left side v is independent of i so that taking maximum with respect to i, we obtain $v \geq \max_{i}\left(\min_{j} DFV(\tilde{g}_{ij})\right)$. Proof of the second part is similar.

c) Theorem 6;

– If player-A possesses a pure optimal strategy i^*, then

$$v = \max_{i}\left(\min_{j} DFV(\tilde{g}_{ij})\right) = \min_{j} DFV(\tilde{g}_{i^*j})$$

– (ii). If player-B possesses a pure optimal strategy j^*, then $v = \min_{j}\left(\max_{i} DFV(\tilde{g}_{ij})\right) = \max_{i} DFV(\tilde{g}_{ij^*})$

Proof:

$v = \max_{\xi}\min_{j} E(\xi, j) = \min_{j} E(e_{i^*}^m, j)$ as $\xi^* = e_{i^*}^m$ is optimal. Proof of the rest is similar.

G. Solution Of Fuzzy Matrix Game:

Let us consider a 2×2 Matrix game whose fuzzy payoff matrix $\tilde{G}$ is given by $\tilde{G} = \begin{bmatrix} \tilde{g}_{11} & \tilde{g}_{12} \\ \tilde{g}_{21} & \tilde{g}_{22} \end{bmatrix}$

If $\tilde{G}$ has no saddle point, then the value of the game is

$$v = \frac{DFV(\tilde{g}_{11})DFV(\tilde{g}_{22}) - DFV(\tilde{g}_{12})DFV(\tilde{g}_{21})}{(DFV(\tilde{g}_{11}) + DFV(\tilde{g}_{22})) - (DFV(\tilde{g}_{12}) + DFV(\tilde{g}_{21}))}$$

,provided

$(DFV(\tilde{g}_{11}) + DFV(\tilde{g}_{22})) - (DFV(\tilde{g}_{12}) + DFV(\tilde{g}_{21})) \neq 0$.

It can be proved that $(DFV(\tilde{g}_{11}) + DFV(\tilde{g}_{22})) - (DFV(\tilde{g}_{12}) + DFV(\tilde{g}_{21})) = 0$ implies that $\tilde{G}$ has a saddle point.

IV. SOME DEFUZZIFICATION TECHNIQUES

Defuzzification is the process of producing a quantifiable result in fuzzy logic, given the fuzzy sets and the corresponding degrees of membership. There are several Defuzzification techniques available in the existing literature. Here two general techniques are taken and compared. The techniques are Centre of Area of Fuzzy Number (COA of Fuzzy Number) and Bisector of Area of Fuzzy Number (BOA of Fuzzy Number).

A. Centre of Area Of Fuzzy Number (COA Of Fuzzy Number):

This defuzzification can be expressed as $x_{COA} = \frac{\int_x x\mu_A(x)dx}{\int_x \mu_A(x)dx}$ where x_{COA} is the crisp output, $\mu_A(x)$ is the membership function corresponding to the fuzzy number and x is the output variable. This method is also known as center of gravity or centroid defuzzification method.

Defuzzified values for Centre of Area of Fuzzy Number for trapezoidal fuzzy fuzzy number (TrFN) is

$$x_{COA} = \frac{\left[(a+b+c+d) - \frac{(cd-ab)}{(d+c-b-a)}\right]}{3}.$$

Defuzzified values for Centre of Area of Fuzzy Number for trapezoidal parabolic fuzzy number (TrPFN) is

$$x_{COA} = \frac{(3d^2 - 3a^2 + c^2 - b^2 - 2ab + 2cd)}{4(2d - 2a - b + c)}.$$

B. Bisector Of Area Of Fuzzy Number (BOA Of Fuzzy Number):

The bisector of area is the vertical line that divides the region into two sub-regions of equal area. The formula for x_{COA} is given by $\int_a^{x_{BOA}} \mu_A(x)dx = \int_{x_{BOA}}^{d} \mu_A(x)dx$ .It is sometimes, but not always coincident with the centroid line.

Defuzzified values for Bisector of Area of Fuzzy Number for trapezoidal fuzzy fuzzy number (TrFN) is $x_{BOA} = \frac{(a+b+c+d)}{4}$.

Defuzzified values for Bisector of Area of Fuzzy Number for trapezoidal parabolic fuzzy number (TrPFN) is $x_{BOA} = \frac{(2a+2d+b+c)}{6}$.

V. NUMERICAL EXAMPLE

Suppose that there are two firms X and Y to enhance the market share of a new product by competing in advertising. The two firms are considering two different strategies to increase market share: strategy I (adv. by pamphlets), II (adv. by road banners). Here it is assumed that the targeted market is fixed, i.e. the market share of the one firm increases while the market share of the other firm decreases and also each company puts all its advertisements in one. The above problem may be regarded as matrix game. Namely, the firm X and Y are considered as players X and Y respectively.The marketing research department of firm X establishes the following pay-off matrix

$$\tilde{G} = \begin{bmatrix} (145,150,160,165) & (120,125,135,145) \\ (50,60,70,80) & (140,155,160,165) \end{bmatrix}$$

Where the element in the matrix indicates that the sales amount of the X increase by “about 160” units when the firm X and Y use the strategy I (adv. by pamphlets) simultaneously. The other elements in the matrix can be explained similarly.

Defuzzification Methods	Defuzzified Pay of Matrix	Value of the game
COA - TrFN	$\tilde{G} = \begin{bmatrix} 155 & 131.43 \\ 65 & 154.44 \end{bmatrix}$	$v = 136.23$
COA - TrPFN	$\tilde{G} = \begin{bmatrix} 155 & 131.77 \\ 65 & 154.86 \end{bmatrix}$	$v = 136.35$
BOA - TrFN	$\tilde{G} = \begin{bmatrix} 155 & 131.25 \\ 65 & 155 \end{bmatrix}$	$v = 136.21$
BOA - TrPFN	$\tilde{G} = \begin{bmatrix} 155 & 131.66 \\ 65 & 154.16 \end{bmatrix}$	$v = 136.23$

The computational game value results have been shown in the above table for different parametric values. Due to the choices of decision makers’, the payoff value in a zero sum game might be imprecise rather than precise value. The optimal solution sets, as obtained by the defuzzification approach, are consistent with those obtained by standard existing approach under fuzzy set up. Thus, it can be claimed that the defuzzification approach attempted in this work well to handle the matrix game with fuzzy payoff.

VI. CONCLUSION

In this paper, a method of solving fuzzy game problem using two fuzzy Defuzzification techniques of fuzzy numbers has been considered. A Numerical example is presented to illustrate the proposed methodology. Due to the choices of decision makers’, the payoff value in a zero sum game might be imprecise rather than precise value. This impreciseness may be represented by various ways. In this paper, we have represented this by fuzzy number. Then the fuzzy game problem has been converted into crisp game problem after Defuzzification in which all the payoff values are crisp valued. Here, two Defuzzification techniques have been used to solve the fuzzy game and the corresponding crisp games with their strategies and value of the game have been presented and compared.

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