A Problem Arising in Viscous Fingering Phenomenon in Two Phase Immiscible Flow with Singular Perturbation Technique

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Abstract-- This paper discusses a well-known phenomenon of fingering which occurs in multi immiscible phase flow through homogeneous porous media. Many experiments shows that the instabilities depend on the mobility ratio and are more likely to occur if the mobility ratio is higher than 1. In fact injected fluids, that are more mobile than native fluid can cause harmful instabilities. The governing law of Darcy, governing equation of continuity and certain basic assumptions yields a nonlinear partial differential equation for motion of saturation of injecting fluid. The analytical solution is obtained by using perturbation technique.

Keywords: perturbation, fingering, immiscible, instability

I. INTRODUCTION

This paper describes the phenomenon of fingering which occurs in two phase immiscible flow through homogeneous porous media. In case of thicker medium, the vertical component of the velocities cannot be ignored, and the analysis of the forces acting shows that the interfaces and the fronts are generally encroached. This encroachment occurs on the front at a small scale called fingering and are governed by conditions of instability. Many experiments shows that these instabilities depend, in particular, on the mobility ratio when it is higher than 1. The native fluid can cause harmful instabilities when injected fluids are more mobile. The phenomenon of fingering occurs due to the difference in viscosities of flowing fluids [1], [2], [3], [4]. In the present work, this phenomenon has been considered with capillary pressure. The assumption made is that the individual pressure of the two flowing phases may be replaced by the common mean pressure and the behavior of the fingers is determined by a statistical treatment. The governing law of Darcy, governing equation of continuity and certain basic assumptions yields a nonlinear partial differential equation for motion of saturation of injecting fluid. The analytical solution is obtained by using perturbation technique.

II. STATEMENT OF THE PROBLEM

When a fluid contained in a porous medium is displaced by another fluid of lesser viscosity, instead of regular displacement of the whole front, protuberance may occur which shoot through the porous medium at relatively great speed. This phenomenon is called fingering. We consider, here, that a finite cylindrical piece of homogeneous porous medium of length L, fully saturated with an oil is displaced by injecting water which give rise to fingers. Since the entire oil at the initial boundary is displaced through a small distance due to water injection,

Therefore, it is assumed that complete saturation exists at the initial boundary.

In the statistical treatment of fingers [5] only the average behavior of the two fluids involved is taken into consideration. The treatment of the motion with the introduction of the concept of fictitious relative permeability becomes formally identical to the Buckley-Leverett description of two immiscible fluid flow through porous media. The saturation of injected fluid (S_i) is then defined as an average cross-sectional area occupied by the injected fluid at level x at time t , i.e. $S_i(x,t)$ [6].

III. EQUATION OF MOTION FOR SATURATION

Assuming the validity of Darcy's law, in the case of being investigated the seepage velocity of water v_i and oil v_o may be written as :

$$v_i = -\frac{k_i}{\mu_i} K \frac{\partial p_i}{\partial x} \qquad (1)$$

$$v_o = -\frac{k_o}{\mu_o} K \frac{\partial p_o}{\partial x} \qquad (2)$$

Where K is the permeability of the homogeneous medium, k_i and k_o are relative permeabilities assumed to be functions of water saturation S_i and S_o oil saturation respectively, p_i and p_o are the pressures μ_i and μ_o are viscosities of water and oil respectively.

The equations of continuity are:

$$\phi \frac{\partial S_i}{\partial t} + \frac{\partial v_i}{\partial x} = 0 \qquad (3)$$

$$\phi \frac{\partial S_o}{\partial t} + \frac{\partial v_o}{\partial x} = 0 \qquad (4)$$

Where ϕ is the porosity of the medium. The capillary pressure p_c , defined as the pressure discontinuity of the flowing phases across their common interface, may be written as:

$$p_c = p_o - p_i \qquad (5)$$

As we have considered the porous medium to be fully saturated, it is evident that

$$s_i + s_o = 1 \qquad (6)$$

The equation of motion for saturation can be obtained by substituting the value of v_i from equation (1) into (3), and then eliminating $\frac{\partial p_i}{\partial x}$ from equation (5), we get

$$\emptyset \frac{\partial S_i}{\partial t} = \frac{\partial}{\partial x}\left[\frac{k_i}{\mu_i} K \left\{\frac{\partial p_o}{\partial x} - \frac{\partial p_c}{\partial x}\right\}\right] \qquad (7)$$

Now, substituting equation (2) into (4), and then using equations (6) and (7), we get

$$\frac{\partial}{\partial x}\left[\left\{\frac{k_i}{\mu_i} + \frac{k_o}{\mu_o}\right\} K \frac{\partial p_o}{\partial x} - \frac{k_i}{\mu_i} K \frac{\partial p_c}{\partial x}\right] = 0 \qquad (8)$$

Integrating equation (8) with respect to x, and simplifying the result, we get

$$\frac{\partial p_o}{\partial x} = -\frac{A}{\left\{\frac{k_i}{\mu_i} + \frac{k_o}{\mu_o}\right\}} + \frac{\frac{\partial p_c}{\partial x}}{1 + \frac{k_o \mu_i}{k_i \mu_o}} \qquad (9)$$

Where A is a constant of integration (Here, negative sign is considered for our convenience). Substituting the value of $\frac{\partial p_o}{\partial x}$ from (9) into the equation (7), we get

$$\emptyset \frac{\partial S_i}{\partial t} + \frac{\partial}{\partial x}\left[\frac{k_o}{\mu_o} K \frac{\partial p_c}{\partial x} \frac{1}{1+\frac{k_o \mu_i}{k_i \mu_o}} + \frac{A}{1+\frac{k_o \mu_i}{k_i \mu_o}}\right] = 0 \quad (10)$$

The value of the pressure of oil can be written as [7]

$$p_o = \bar{p} + \frac{p_c}{2}, \quad \bar{p} = \frac{p_c + p_i}{2} \quad (11)$$

Where $\bar{p}$ is the mean pressure constant? This gives the value of A from (9) as

$$A = \left\{\frac{k_i}{\mu_i} - \frac{k_o}{\mu_o}\right\} \frac{K}{2} \frac{\partial p_c}{\partial x}$$

And thus from (10), we get

$$\emptyset \frac{\partial S_i}{\partial t} + \frac{\partial}{\partial x}\left[K \frac{k_i}{2\mu_i} \frac{\partial p_c}{\partial x}\right] = 0 \quad (12)$$

For definiteness, we assume standard form of an analytical expression [5] for the relationship between the relative permeability, phase saturation and capillary pressure as

$$k_i = S_i, \quad k_o = S_o = 1 - S_i, \quad and \quad p_c = -\beta S_i \quad (11A)$$

We consider negative sign which shows the direction of saturation of water is opposite to capillary pressure. For definiteness, we consider β small parameter.

Combining equations (11A) and (12), we get

$$\emptyset \frac{\partial S_i}{\partial t} - \frac{\beta K}{4\mu_i} \frac{\partial^2 S_i{}^2}{\partial x^2} = 0 \quad (13)$$

This is a nonlinear partial differential equation of motion for the saturation of the injected fluid through the homogeneous porous medium. Here β is multiplied to the highest derivative, thus the problem is a singular perturbation problem.

IV. SOLUTION OF THE PROBLEM

To obtain the solution in the asymptotic series form in terms of perturbation parameter the boundary conditions are

$$S_i(0,t) = S_{w0} \quad and \quad S_i(L,t) = S_{wl} \quad (14)$$

Where S_{w0} and S_{wl} are respectively the saturations at *x=0* and *x=L*. Also we assume that there is no flow across the face x = L and thus

$$\frac{\partial}{\partial x} S_i(L,t) = 0 \quad (15)$$

Choosing the dimensionless variables $X = \frac{x}{L}$ and $T = \frac{K}{4\mu_i L^2 P} t$ the equation (13) together with (14) and (15) becomes

$$\frac{\partial S_i}{\partial T} - \beta \frac{\partial^2 S_i{}^2}{\partial X^2} = 0 \quad (16)$$

And

$$S_i(0,T) = S_{w0} \quad and \quad S_i(L,T) = S_{wl} \quad , \quad \frac{\partial}{\partial X} S_i(1,T) = 0 \quad (17)$$

To solve this problem we use Birkhoff's technique of one parameter group transformation to get,

$$\beta\left[FF'' + F'^2\right] + \frac{B}{2}\eta F' - AF = 0 \quad (18)$$

Where

$$\eta = \frac{X}{T^B}, \quad F(\eta) = \frac{S_i(X,T)}{T^{2A}}$$

. We choose A > 0.5 and B > 0.

Also

$$F(0) = S_{w0},$$

$$F(\alpha) = S_{wl} \quad where \quad \alpha = 1/T^B, \quad and \quad F'(\alpha) = 0 \quad (19)$$

Solving the nonlinear equation (18) together with (19) by using singular perturbation technique [9] Since the coefficient of F'>0 in $(0,\alpha)$, the non uniformity occurs near $\eta = 0$ to describe F in the region, we need a stretching transformation $\zeta = \eta/\beta$. We assume that

$$F(\eta,\beta) = H(\eta,\beta) + G(\zeta,\beta) = H_0(\eta) + G_0(\zeta) + \beta[H_1(\eta) + G_1(\zeta)] + \cdots \quad (20)$$

Where $G(\zeta,\beta)$ is negligible outside the inner region. So that the outer expansion is given by

$$F^0(\eta,\beta) = H(\eta,\beta) = H_0(\eta) + \beta H_1(\eta) + \cdots \quad (21)$$

Since $= \beta\zeta$, the inner expansion can be written as

$$F^i(\beta\zeta,\beta) = H_0(0) + G_0(\zeta) + \beta[H_0'(0)\zeta + H_1(0) + G_1(\zeta)] + \cdots \quad (22)$$

Since $G(\zeta,\beta)$ is assumed to be negligible outside the boundary layer, $H(\eta,\beta)$ satisfies the boundary condition $F(\alpha) = S_{wl}$. Hence $H_0(\alpha) = S_{wl}$, $H_n(\alpha 0 = 0 \quad for \quad n \geq 1$ (23)

The boundary condition $F(0) = S_{w0}$ is satisfied by H+G. So that

$$H_0(0) + G_0(0) = S_{w0} \quad and \quad H_n(0) + G_n(0) = 0 \quad for \quad n \geq 1 \quad (24)$$

Hence the equations governing H_n can be obtained as

$$BH_0' - \frac{2A}{\eta} H_0 = 0 \quad and \quad \frac{\eta B}{2} H_1' - AH_1 = -H_0 H_0'' - H_0{}'^2 \quad (25)$$

And the equations governing G_n can be determined in terms of $\zeta = \eta/\beta$ as

$$F \frac{d^2F}{d\zeta^2} + \left\{\frac{dF}{d\zeta}\right\}^2 + \frac{B}{2}\zeta \frac{dF}{d\zeta} - \beta AF = 0 \quad (26)$$

Substituting (22) into (26) keeping ζ fixed and equating the coefficients of like powers of β, we get

$$H_0(0)G_0'' + G_0 G_0'' + G_0{}'^2 = 0 \quad (27)$$

$$[H_0(0) + G_0]G_1'' + 2G_0'G_1' + G_0{}''^{G_1} = -H_0'(0)\zeta G_0'' + H_1(0)G_0'' - 2H_0'(0)G_0' - \frac{B}{2}\zeta G_0' + \quad (28)$$

Now we solve equations (25), (27) and (28) to get

$$H_0(\eta) = S_{wl} T^{2A} \eta^{2A/B} \quad (29)$$

$$G_0(\zeta) = S_{w0} \quad (30)$$

$$H_1(\eta) = A_1(T)\eta^{2A/B}\left\{1 - \frac{1}{T\eta^{1/B}}\right\} \quad (31)$$

Where $A_1(T) = -2S_{wl}{}^2 \frac{A(6A-1)}{B^2} T^{4A+1}$

Which when combined with (24) for *n = 1* gives $G_1(0) = 0$

Substituting for H_0, H_1 and G_0 into (28) and solving the resulting equation subject to $G_1(0) = 0$, we get

$$G_1(\zeta) = \frac{A}{2}\zeta^2 - \left\{\frac{2A}{B\alpha} S_{wl} + A\alpha\beta^{-1}\zeta\right\} \quad (32)$$

Now substituting the values of H_0, H_1, G_0 and G_1 into equation (20) gives the solution of (18) subject to the conditions (19) in terms of perturbation parameter β as follows

$$F(\eta,\beta) = S_{wl} T^{2A} \eta^{2A/B} + S_{w0} + \beta\left[A_1(T)\eta^{\frac{2A}{B}}\left\{1 - \frac{1}{T^{\frac{1}{B}}}\right\} + \left\{\frac{A}{2}\zeta^2 - \left(\frac{2A}{B\alpha} S_{wl} + A\alpha\beta^{-1}\right)\zeta\right\}\right] + \cdots \quad (33)$$

Hence, finally we obtain

$$S_w(X,T) = T^{2A} F\left(\frac{X}{T^B}\right) \quad (34)$$

Where $F\left(\frac{X}{T^B}\right)$ can be obtained from (33) by substituting $\zeta = \eta/\beta \text{ and } \eta = X/T^B$.

V. CONCLUDING REMARKS

Equation (34) is the higher ordered approximate solution for the distribution of fictitious displacing phase saturation, that ia an analytical expression for the average cross-sectional area occupied by the fingers.

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