Fixed Point Theorem for a Pair of Mappings in Complete Digital Metric Space

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Abstract — In this research paper we prove some fixed-point theorems for digital images. Ege and karaca stated and proved Banach contraction principal for digital images. The objective of this note is to establish a unique digital fixed-point theorem for a pair of mapping in complete metric space. Main objective of the research article is to present another generalization of the known Banach contraction mapping principal for digital images in pair of mapping. In a sense our result is an improvement of the result of Dolhare (2018).

Keywords: Digital Image, Digital Metric Space, Fixed Point

I. INTRODUCTION

Fixed point theory plays an important role in functional analysis and it has wide application in differential and integral equations, computer science, game theory, image processing. In recent years there have been many developments in digital topology. Ege and Karaca [6] constructed Lefschetz fixed point theory for digital images is studied the fixed-point properties of digital images, Ege and karaca [5] proved Banach fixed points theorem for digital images and gave an application to image processing.

II. PRELIMINARIES

Let X be a subset of Z^n for a positive integer n where Z^n is the set of lattice points in the n-dimensional Euclidean space and k be represent an adjacency relation for the members of X a digital image consists of (X, k).

Definition 2.1[2] Let l ,n be positive integers, $1 \leq l \leq n$ and two say that two distinct points $p = (p_1, p_2, p_3, \dots \dots \dots \dots, p_n)$ and $q = (q_1, q_2, q \dots \dots \dots \dots, q_n) \in Z^n$ p and q are k_{i-}adjacent if there are at most l indices i such that $|p_i - q_i|$=1 and for all other indices j such that$|p_j - q_j|\neq 1$, $p_j = q_j$.

There are some statements which can be obtained from Definition 2.1:

- Two points p and q in Z are 2-adjacent if $|p - q|$=1.

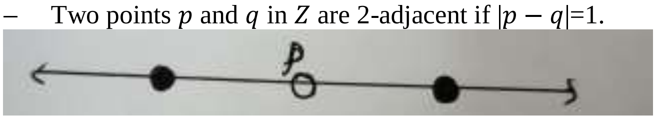

Fig. 1: 2-adjacent

- Two points p and q in Z^2 are 8-adjacent if they are distinct and differ by at most 1 in each coordinate.
- Two points p and q in Z^2 are 4-adjacent if they are 8-adjacent and differ in exactly one coordinate.

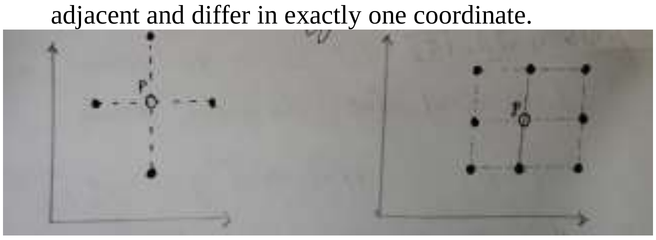

Fig. 2: 4-adjacent and 8- adjacent.

A. *Banach Fixed Point Theorem for Digital Images*

Let (X, k) be a digital image and $f: (X, k) \to (X, k)$ be any (k, k)-continuous function. We say the digital image (X, k) has the fixed-point property [10] if for every (k, k)-continuous map $f: X \to X$ there exists $x \in X$ such that $f(x) = x$ .The fixed-point property is preserved by any digital isomorphism, i.e. it is a topological invariant. Let (X, d, k)denote the digital metric space with k -adjacency where d is usual Euclidean metric for Z^n.

- *Definition 2.2.1*. A sequence $\{x_n\}$ of points of a digital metric space(X, d, k) is a Cauchy sequence if for all ε> 0, there exists $\alpha \in N$ such that for all $n, m > \alpha$, then
$$d(x_n, x_m) < \varepsilon.$$
- *Definition 2.2.2*. A sequence$\{x_n\}$ of points of a digital metric space(X, d, k) converges to a limit $a \in X$ if for all ε> 0, there exists $\alpha \in N$ such that for all $n, m > \alpha$, then
$$d(x_n, a) < \varepsilon.$$
- *Definition 2.2.3*. A digital metric space(X, d, k) is a complete digital metric space if any Cauchy sequence $\{x_n\}$ of points of(X, d, k) converges to a point a of (X, d, k) .
- *Definition 2.2.4*. Let(X, k) be any digital image. A function $f: (X, k) \to (X, k)$is called right- continuous if
$$f(a) = \lim_{x \to a^+} f(x).$$
- *Definition 2.2.5*. Let(X, d, k) be any digital metric space and $f: (X, d, k) \to (X, d, k)$ be a self digital map. If there exists $\lambda \in$ (0,1)such that for all $x, y \in X$,
$$d\big(f(x), f(y)\big) \leq \lambda d(x, y),$$

Then f is called a digital contraction map.

Ege and karaca [5] proved the following

- *Proposition 2.2.6*. Every digital contraction map is digitally continuous.

Proof:-

Let (X, d, k) be a digital metric space and $f: X \to X$ be a digital contraction map. Pick $a \in X$ and let $\varepsilon > 0$ .Let $\delta = \varepsilon$. Then if $d(a, b) < \delta$, we have
$$d(f(a), f(b)) \leq \lambda d(a, b) < \lambda\varepsilon < \varepsilon$$
Where $\lambda \in (0,1)$ for all $a, b \in X$.Then f is a(k, k)-continuous function.

- Theorem 2.2.7. (Banach contraction principal)

Let (X, d, k) be a complete digital metric space which has a usual Euclidean metric in Z^n . Let $f: X \to X$ be a digital contraction map. Then f has a unique fixed point, i.e. there exists a unique $c \in X$ such that $f(c) = c$.

III. MAIN RESULT

Bhagwat and Singh (1986) [1] have extended the result of Das and Gupta (1979) [3] and prove the following theorem.

Let T_1 *and* T_2 be two continuous self-mapping of a metric space (X, d) such that

$$d\,(T_1x, T_2\,y) \le \frac{d\,(x\,,T_1x)\,d(x\,,T_2\,y) + d\,(y\,,T_2y)\,d(\,y,T_1x)}{d\,(x,T_2y) + d(\,y,T_1x)}$$

for all $x, y\ in\ X$.

Now we generalized theorem triple metric space and prove the theorem.

A. *Theorem*

Let (X, d, k) be a complete digital metric space. Let T and S be continuous self-mapping as T and $S : (X\ ,d\ ,k) \to (X\ ,d\ ,k)$ satisfies ①

$$d\,(Tx\ ,Sy) \le \frac{d(x\ ,Tx)\,d(x\ ,Sy)+d\,(y\ ,Sy)\,d(\,y,Tx)}{d\,(x,S\,y)\,+d\,(y,Tx)} \quad ②$$

$\forall, x\ ,y\ in\ X\ .\ If\ for\ some\ x_0 \in X, the\ sequence\ \{x_n\} of\ elements\ x_n$
$where\ x_{\ 2n+1} = T\,x_{2n}\ ,\ \ x_{\ 2n+2} = Sx_{\ 2n+1}$
$Has\ \ a\ convergent\ subsequence\ \{x_{nk}\} convergent\ to\ a\ point\ a \in X, then\ a\ is\ a\ unique$
$fixed\ point\ of\ T\ and\ S\ in\ (X, d, k).$

Proof: - Let x_0 be an arbitrary but fixed element in (X, d, k) defines a sequence of iteration $\{x_n\}_{n=1}^{\infty}$ in X by,

$$x_1 = Tx_0, \qquad x_2 = Sx_1,$$
$$x_3 = Tx_2, \qquad x_4 = S\,x_3\,,$$
$$x_5 = T\,x_4, \qquad x_6 = S\,x_5, \_\_\_\ \_\_\_\ \_\_\_\ \_\_\_$$
$$\_\_\_\ \_\_\_\ \_\_\_\ \_\_\_$$
$$x_{2n+1} = T\,x_{2n}, \qquad x_{2n+2} = S\,x_{2n+1}$$
$$0 \le d\,(x_{2n+1}, x_{2n+2}) = d\,(T\,x_{2n}\,, S\,x_{2n+1})$$
$$\le \frac{d(\,x_{2n}\,,Tx_{2n}\,)d(x_{2n}\,,Sx_{2n+1}\,) + d(x_{2n+1}\,,Sx_{2n+1}\,)d(\,x_{2n+1}\,,Tx_{2n}\,)}{d(x_{2n}\,,Sx_{2n+1}\,) + d(x_{2n+1}\,,Tx_{2n}\,)}$$
$$\le \frac{\mathrm{d}(\,x_{2n}\,,x_{2n+1}\,)\mathrm{d}(x_{2n}\,,x_{2n+2}) + \mathrm{d}(x_{2n+1}\,,x_{2n+2}\,)\mathrm{d}(\,x_{2n+1}, x_{2n+1})}{\mathrm{d}(x_{2n}\,,x_{2n+2}\,) + \mathrm{d}(x_{2n+1}, x_{2n+1})}$$
$$\le \frac{d(\,x_{2n}\,,x_{2n+1}\,)\,d(\,x_{2n}, x_{2n+2})}{d(x_{2n}\,,x_{2n+2}\,)}$$
$$\therefore\ d(\,x_{2n+1}, x_{2n+2}) \le d(x_{2n}\,,x_{2n+1})$$

Continuing in this way, we obtain

$$d(\,x_{2n+1}, x_{2n+2}) \le d(x_{2n}\,,x_{2n+1}\,) \le \_\ \ \_\_\_\_\_\_ \le d(x_0\,,x_1\,)$$

So, we have a monotonic sequence of positive real number.
Let this monotonic sequence converges to real number l.
Since $\{x_n\}$ has a convergent sequence $\{\,x_{nk}\,\}$ in X which converges to some $a \in X$,
Therefore $\lim_{n\to\infty} x_{2nk} = a$

Then we have to show that a is a unique fixed point of T and S. If possible, we assume that $a \neq Ta$

Now,

$$d(a\,,Ta) = d(\,lim\ x_{2nk}, T\ lim\ x_{2nk}\,)$$
$$= lim\ d(x_{2nk}, T\,x_{2nk}\,)$$
$$= lim\ d(\,x_{2nk}, \qquad x_{2nk+1}\,)$$
$$= lim\ d(x_{2nk+1}, \qquad x_{2nk+2}\,)$$
$$= lim\ d(Tx_{2nk}, \qquad Sx_{2nk}\,)$$
$$= d(\,lim\ Tx_{2nk}, \qquad S\ lim\ \ Tx_{2nk})$$
$$= d(Ta, STa) \quad ③$$

But

$$d(Ta, STa) \le \frac{d(a,Ta)\,d(a,STa) + d(Ta,STa)\,d(Ta,Ta)}{\mathrm{d(a,STa)} + \mathrm{d(Ta,Ta)}}$$
$$\le \frac{d(a,Ta)d(a,STa)}{d(a,STa)}$$
$$\le d(a,Ta) \quad ④$$

From ③ and ④ we have
$d(a,Ta) = d(Ta,STa) \le d(a,Ta)$
Which is contradiction
Therefore $a = Ta$ i.e. a is a fixed point of T.
In the same way let $a \neq Sa$, we get
$d(a,Sa) = d(Ta,STa) \le d(a,Sa)$
which is a contradiction.
Hence a is a fixed point S also,thus

$$Ta = a = Sa \quad ⑤$$

Let $a = a'$,then

$$d(a,a') = d(Ta,Sa') \leq \frac{d(a,Ta)d(a,STa')+d(a',Sa')d(a',Ta)}{d(a,Sa')+d(a',a)} \leq 0$$

A contradiction, therefore a is unique fixed point of T and S.
This completes the proof of the theorem.

IV. REMARKS:

Our result is the extension of Bhagwat and Singh in digital metric space

REFERENCES:

[1] O Ege and I Karaca. “Lefschetz fixed point theorem for digital images”. *Fixed point theory Appl*, 2002.
[2] O Ege and I Karaca. “Banach fixed point theorem for digital images”. *J. Nonlinear. Sci.Appl*, vol. 8, 237–245, 2015.
[3] L Boxer. “A classical construction for the digital fundamental group”. *J. Math. Imaging Vis*, vol. 10, 51–62, 1999.
[4] H K Pathak. “A note on fixed point theorems of Rao and Rao”. *Bull. Cal. Math*, vol. 79, 267–273, 1987.
[5] N Bhagwat and B Singh. “Fixed points of a pair of mappings”. *Indian J. Pure A Math*, vol. 17, no. 8, 994–997, 1986.
[6] B K Das and S Gupta. *Indian J. Pure Appl, Math*, vol. 10, 886–889, 1979.