Estimating Reliability under Linex Loss of Weibull Model using Bayesian Lindley Approximation

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Abstract — This paper describes the unknown parameter, reliability and hazard function of the Weibull distribution-based on failure censored data. The estimation of parameter of the Weibull distribution is considered with a Natural Conjugate Gamma Prior and Linex Loss Function. Lindley's approximation is used to obtain Approximate Bayes estimator of Reliability function. The result from Bayesian method is used to compare with Bayes and Maximum likelihood estimate (MLE) methods. The simulation shows that the results from Bayes is Robust for Approximate Bayesian method than MLE in terms of mean square error (MSE).

Keywords: Linex Loss Function, Maximum Likelihood Estimation, Bayesian Estimation, Reliability, Lindley Approximation, Weibull Distribution

I. Introduction

For Bayesian inference, a frequent choice of loss function is a Squared Error loss function. However, Bayesian estimation under this loss function is not frequently discussed, perhaps, because the estimators under symmetric and asymmetric loss function involve integral expressions, which are not analytically solvable. Therefore, one has to use the numerical techniques or certain approximation methods for the solution. One of the most suitable loss function Precautionary loss functions, which is asymmetrical. Lindley's approximation is the method suitable for solving such problems. There has been a significant amount of research done in statistical inference of several distributions.

The Weibull distribution was introduced by the Swedish physicist Weibull [1959], it has been used in many different fields like material science, engineering, physics, chemistry, meteorology, medicine, pharmacy, economics and business, quality control, biology, geology and geography. The two parameters Weibull distribution is one of the most widely used lifetime models in reliability and survival analysis because of its various shapes of the probability density function(pdf) and its convenient representation of the Reliability and Hazard Function. The estimation of its parameters has been discussed by a number of authors.[Zakerzadeh and Jafari [2014], Doostparast [2006], Modarress, Kaminskiy and Krivtsov [2006], Sun and Berger[2008] and Kundu and Joarder [2006] and Kundu [2007]]. The properties of the Weibull distribution are best described in terms of the hazard function. This tells us how likely something is to fail given that it has survived so far. Weibull distribution has also been extensively used in life testing and reliability probability problems. Estimation and properties of the Weibull distribution is studied by many author's[Kao (1959)].

The probability density function reliability and hazard rate functions of Weibull distribution are given respectively as

$$f(x) = p\theta x^{(p-1)} \exp(-\theta\, x^{P}) \quad ; \quad x,\theta,p > 0 \qquad (1.1)$$

$$R(t) = exp(-\theta t^{p}) \quad ; \quad t > 0 \qquad (1.2)$$

$$H(t) = p\theta t^{(p-1)} \quad ; \quad t > 0 \qquad (1.3)$$

Where $'\theta'$ is the scale and 'p' is shape parameters.

The most widely used loss function in estimation problems is quadratic loss function given as $L(\hat{\theta},\theta) = k(\hat{\theta} - \theta)^2$ where $\hat{\theta}$ is the estimate of θ, the loss function is called quadratic weighed loss function if k=1, we have

$$L(\hat{\theta},\theta) = (\hat{\theta} - \theta)^2 \qquad (1.4)$$

known as squared error loss function (SELF). This loss function is symmetrical because it associates the equal importance to the losses due to overestimation and under estimation with equal magnitudes however in some estimation problems such an assumption may be inappropriate. Overestimation may be more serious than underestimation or Vice-versa Ferguson(1985). Canfield (1970), Basu and Ebrabimi(1991). Zellner (1986) Soliman (2001) derived and discussed the properties of varian's (1975) asymmetric loss function for a number of distributions. Such as a loss function is derived as

$$L(\Delta) = b\exp(a\Delta) - c\Delta - b; \quad \Delta = (\hat{\theta} - \theta), \qquad (1.5)$$

and $a, c \neq 0,\ b > 0$

For a minimum to exist at Δ=0.

$$\left[\frac{\partial}{\partial\Delta} L(\Delta)\right]_{\Delta=0} = 0 = ab - c$$

And we have a two parameter loss function

$$L(\Delta) = b\,[\exp(a\Delta) - a\Delta - 1];\ b>0, a>0, \qquad (1.6)$$

The sign and magnitude of 'a' represents the direction and degree of symmetry respectively, when a>0, then overestimation is more serious than underestimation and vice-versa. For 'a' closed to zero the LINEX loss is approximately squared error loss and therefore almost symmetric.

The posterior expectation of LINEX loss function in eqn.(1.5) is

$$E_\pi\left(L(\hat{\theta} - \theta)\right) \propto e^{a\hat{\theta}} E_\pi(e^{-a\theta}) - a\left(\hat{\theta} - E_\pi(\theta)\right) - 1 \qquad (1.7)$$

where E_π is the posterior expectation with respect to posterior density of θ.

The Bayes Estimator $\hat{\theta}_{BL}$ of θ under LINEX loss function is the value which minimizes eqn.(1.6) is

$$\hat{\theta}_{BL} = = -\frac{1}{a} log\left(E_\pi(e^{-a\theta})\right); \qquad (1.8)$$

provided that $E_\pi(e^{-a\theta})$ exists and is finite. [Calabria and Pulcini(1994,1996)].

In a Bayesian setup, the unknown parameter is viewed as random variable. The uncertainty about the true value of parameter is expressed by a prior distribution. The parametric inference is made using the posterior distribution which is obtained by incorporating the observed data into the prior distribution using the Bayes theorem, the first theorem of inference. Hence, we update the prior distribution in the light of observed data. Thus, the uncertainty about the parameter prior to the experiment is represented by the prior distribution and the same after the experiment is represented by the posterior distribution. The various statistical models are considered.

The chapter deals with the methods to obtain the approximate Bayes estimators of the Weibull distribution by using Lindley approximation technique for type-II censored samples. A bivariate prior density for the parameters, squared error Loss function (SELF) and Linex Loss function are used to obtain the approximate Bayes Estimators. A statistical software R is used for numerical calculations for different approximate Bayes estimators and their relative mean squared errors by preparing programs to present the statistical properties of the estimators

A. The Estimators

Let $x_1, x_2, \dots\dots\dots x_n$ be the life times of 'n' items that are put on test for their lives, follow a weibull distribution with density given in equation (3.1.1) . The failure times are recorded as they occur until a fixed number 'r' of times failed. Let $= (x_{(1)}, x_{(2)}, \dots\dots\dots\dots\dots, x_{(n)})$, where $x_{(i)}$ is the life time of the i[th] item . Since remaining (n-r) items yet not failed thus have life times greater than $x_{(r)}$.

The likelihood function can be written as

$$L(x|\theta,p) = \frac{n!}{(n-r)!}(\mathrm{p}\theta)^r \prod_{i=1}^{r} x_i^{(p-1)} \exp(-\delta\theta), \qquad (2.1)$$

Where

$$\delta = \sum_{i=1}^{r} x_i^p + (n-r)x_r^p$$

The logarithm of the likelihood function is

$$\log L(x|\theta,p) \propto r\, log\, p + r \log\, \theta + (p-1)\sum_{i=1}^{r} \log x_i - \delta\theta, \qquad (2.2)$$

assuming that 'p' is known, the maximum likelihood estimator $\hat{\theta}_{ML}$ of θ can be obtain by using equation (2.2) as

$$\hat{\theta}_{ML} = r/\delta \qquad (2.3)$$

If both the parameters p and θ are unknown their MLE's $\hat{p}_{ML}$ and $\hat{\theta}_{ML}$ can be obtained by solving the following equation

$$\frac{\delta}{\delta\theta}\log L = \frac{r}{\theta} - \delta = 0\ , \qquad (2.4a)$$

$$\frac{\delta \log L}{\delta P} = \frac{r}{P} + \sum_{i=1}^{r} \log x_i - \theta\delta_1 = 0, \qquad (2.4b)$$

Where

$\delta_1 = \sum_{i=1}^{r} x_i^P \log x_i + (n-r)x_r^P \log x_r$, eliminating θ between the two equations of (2.4) and simplifying we get

$$\hat{p}_{ML} = \frac{r}{\delta^*} \qquad (2.5)$$

Where $\delta^* = \left[\frac{r\delta_1}{\delta} - \sum_{i=1}^{r} log x_i\right]$

Equation (2.5) may be solved for Newton- Raphson or any suitable iterative Method and this value is substituted in equation (2.4b) by replacing with p get $\hat{p}$ as

$$\hat{\theta}_{ML} = \frac{\frac{r}{\hat{p}_{ML}} + \sum_{i=1}^{r} log x_i}{\sum_{i=1}^{r} x_i^{\hat{p}_{ML}}\ log x_i + (\mathrm{n-r})x_r^{\hat{p}_{ML}}\ log x_r}, \qquad (2.6)$$

The MLE's of R(t) and H(t) are given respectively by equation (1.2) and (1.3) after replacing θ and p by $\hat{\theta}_{ML}$ and $\hat{p}_{ML}$.

Bayes Estimater of θ when shape Parameter P is known :

If p is known assume gamma prior $\gamma(\alpha,\beta)$ as cojugate prior for θ as

$$g(\theta|\underline{x}) = \frac{\beta^\alpha}{\Gamma\alpha}(\theta)^{(\alpha+1)} \exp(-\beta\theta)\,; (\alpha,\beta) > 0, \theta > 0\ , \qquad (3.1)$$

The posterior distribution of θ using equation (2.1) and (3.1) we get

$$h(\theta|\underline{x}) = \frac{(\delta+\beta)^{r+\alpha}}{\Gamma(r+\alpha)}(\theta)^{(r+\alpha-1)} \exp\big(-\theta(\delta+\beta)\big)\,, \qquad (3.2)$$

Under squared error loss function, the Bayes estimator $\hat{\theta}_{BS}$, is the posterior mean given by

$$\hat{\theta}_{BS} = \frac{(r+\alpha)}{(\delta+\beta)} \qquad (3.3)$$

Under linex Loss Function, the Bayes estimator $\hat{\theta}_{BL}$ of θ using (1.7) and (3.2) given by

$$\hat{\theta}_{BL} = \frac{(r+\alpha)}{a}\ log\left[1 + \frac{a}{(\delta+\beta)}\right] \qquad (3.4)$$

B. Bayes Estimator of R(t)

The posterior distribution of R using equation (1.2) and (3.2), is given as

$$h(R|t) = \frac{[\ \mathrm{c}\,(\delta+\beta)]^{(r+\alpha)}}{\Gamma(r+\alpha)}(-\log R)^{(r+\alpha-1)} R^{(\ \mathrm{c}\,(\delta+\beta)\,-\,1)} dR; \qquad (3.6)$$

Where $c = t^{-p}$

The Bayes estimator of R(t) under squared error loss function using (3.6) is given by

$$\hat{R}_{BS} = \left[1 + \frac{1}{\mathrm{c}\,(\delta+\beta)}\right]^{-(r+\alpha)}; \qquad (3.7)$$

The Bayes estimator of R(t) under Linex loss function

$$\hat{R}_{BL} = \left(-\frac{1}{\mathrm{a}}\right)\ log\left[\sum_{k=0}^{\infty} \frac{(-a)^k}{k!}\left(1 + \frac{k}{c((\delta+\beta))}\right)^{-(r+\alpha)}\right]; \qquad (3.8)$$

C. The Bayes estimators with $\boldsymbol{\theta}$ and p unknown:

The joint prior density of θ and p is given by

$G(\theta|p) = g_1(\theta|p).\, g_2(p)$

$$G(\theta|p) = \frac{1}{\lambda\Gamma\xi} p^{-\xi}\theta^{(\xi-1)}.\exp\left[-\left(\frac{\theta}{p} + \frac{p}{\lambda}\right)\right]\ ; (\theta,p,\lambda,\xi) > 0, \qquad (4.1)$$

where

$$g_1(\theta|p) = \frac{1}{\Gamma\xi} p^{-\xi}\ \theta^{(\xi-1)}.\exp\left[-\frac{\theta}{p}\right]; \qquad (4.2)$$

And

$$g_2(p) = \frac{1}{\lambda}\exp\left(-\frac{p}{\lambda}\right)\quad ; \qquad (4.3)$$

The joint posterior density of θ and p is

$h^*(\theta,p|\underline{x}) =$

$$\frac{\frac{1}{\lambda\Gamma\xi}p^{-\lambda}\theta^{(\xi+1)} \exp\left[-\left\{\frac{\theta}{p}+\frac{p}{\lambda}\right\}\right](\mathrm{p}\theta)^r \prod_{i=1}^{r} x_i^{(p-1)} e^{-p\theta}}{\iint \frac{1}{\lambda\Gamma\xi}p^{(r-\xi)}\theta^{(r+\xi+1)} \prod_{i=1}^{r} x_i^{(p-1)}.\exp\left[-\left\{\frac{\theta}{p}+\frac{p}{\lambda}+\mathrm{p}\theta\right\}\right] d\theta dp}; \qquad (4.4)$$

D. Approximate Bayes Estimators

The Bayes estimators of a function $\mu = \mu(\theta,p)$ of the unknown parameter θ and p under squared error loss is the posterior mean

$$\hat{\mu}_{ABS} = E(\mu|\underline{x}) = \frac{\iint \mu(\theta,p)G(\theta,p|\underline{x})d\theta dp}{\iint G(\theta,p|\underline{x}).d\theta.dp}\ ; \qquad (4.5)$$

To evaluate (4.5) consider the method of Lindley approximation(Lindley(1982))

$$E(\mu(\theta,p)|\underline{x}) = \frac{\int \mu(\theta).e^{(l(\theta)+\rho(\theta))}d\theta}{\int e^{(l(\theta)+\rho(\theta))}d\theta}; \qquad (4.6)$$

Where $(\theta) = \log g(\theta)$, and $g(\theta)$ is an arbitrary function of θ and $l(\theta)$ is the logarithm likelihood function

The Lindley approximation for two parameter is given by

$E(\hat{\mu}(\theta,p)|\underline{x}) = \mu(\theta,p) + \frac{A}{2} + \rho_1 A_{12} + \rho_2 A_{21} + \frac{1}{2}[l_{30}B_{12} + l_{21}C_{12} + l_{12}C_{21} + l_{03}B_{21}],$ (4.7)

Where

$A = \sum_1^2 \sum_1^2 \mu_{ij}\sigma_{ij}$; (3.4.7a) , $l_{\eta\epsilon} = (\delta^{(\eta+\epsilon)} l | \delta\theta_1^\eta \delta\theta_2^\epsilon)$;

where $(\eta+\epsilon) = 3$ for $i,j = 12$ $\rho_i = (\delta\rho|\delta\theta_i)$;

$\mu_i = \frac{\delta\mu}{\delta\theta_i}$; (3.4.7d) , $\mu_{ij} = \frac{\delta^2\mu}{\delta\theta_i\delta\theta_j}$; $\forall i \neq j$;

$A_{ij} = \mu_i\sigma_{ij} + \mu_j\sigma_{ji}$; , $B_{ij} = (\mu_i\sigma_{ii} + \mu_j\sigma_{ij})\sigma_{ii}$; ,

$C_{ij} = 3\mu_i\sigma_{ii}\sigma_{ij} + \mu_j(\sigma_{ii}\sigma_{jj} + 2\sigma_{ij}^2)$;

Where σ_{ij} is the (i,j)th element of the inverse of matrix $\{-l_{jj}\}; i,j = 1,2$ s.t. $l_{ij} = \frac{\delta^2 l}{\delta\theta_i\delta\theta_j}$.

All the function in above equations are evaluated at MLE of (θ_1,θ_2).In our case $(\theta_1\theta_2) = (\theta,p)$; $So\ \mu(\theta) = \mu(\theta,p)$

To apply Lindley approximation (4.5), we first obtain σ_{ij} , elements of the inverse of $\{-l_{jj}\}; i,j = 1,2$,which can be shown to be

$$\sigma_{11} = \frac{M}{D},\ \sigma_{12} = \sigma_{21} = \frac{\delta_1}{D}, \sigma_{22} = \frac{r}{D\,\theta^2},$$ (4.8a)

Where $M = \left(\frac{r}{p^2} + \theta\delta_2\right);\ D = \left[\frac{r}{\theta^2}\left(\frac{r}{p^2} + \theta^2\delta_2\right)\right]$; (4.8b)

$\delta_2 = \sum_{i=1}^{r} x_i^p (logx_i)^2 + (n-r)x_r^p(logx_r)^2$; (4.8c)

To evaluate ρ_i , take the joint prior $G(\theta|p)$

$G(\theta|p) = \frac{1}{\lambda\Gamma\xi} p^{-\xi}\theta^{(\xi-1)}.\exp\left[\left\{-\frac{\theta}{p} + \frac{p}{\lambda}\right\}\right]$; $(\theta,p,\lambda,\xi) > 0$, (4.9)

$\Rightarrow \rho = \log[G(\theta|p)]$

$$= constant - \xi logp - (\xi-1)log\theta - \frac{\theta}{p} - \frac{p}{\lambda}$$

Therefore

$$\rho_1 = \frac{\partial\rho}{\partial\theta} = \frac{(\xi-1)\theta}{\theta} - \frac{1}{p};$$ (4.9a)

and

$$\rho_2 = \frac{\theta}{p^2} - \frac{1}{\lambda} - \frac{\xi}{p};$$ (4.9b)

Further more

$$l_{21} = 0\ ; l_{12} = -\delta_2\ ; l_{03} = \frac{2r}{p^3} - \theta\delta_3;$$ (4.9c)

$$and\ l_{30} = \frac{2r}{\theta^3};$$ (4.9d)

Where $\delta_3 = \sum_{i=1}^{r} x_i^p (logx_i)^3 + (n-r)x_r^p(logx_r)^3$

By substituting above values in eqn. (3.4.7), yields the Bayes estimator under SELF using Lindley approximation denoted by $\hat{\mu}_{ABS}$

$$\hat{\mu}_{ABS} = E(\mu(\theta,p)) = \mu(\theta,p) + Q + \mu_1 Q_1 + \mu_2 Q_2;$$ (4.10)

Where $Q = \frac{1}{2}[\mu_{11}\sigma_{11} + \mu_{21}\sigma_{21} + \mu_{12}\sigma_{12} + \mu_{22}\sigma_{22}]$; (4.10a)

$$Q_1 = \frac{1}{\theta^2 D^2}\left[\frac{M\theta D}{p}(p(\xi-1)-1) + \frac{\theta^2\delta_1 D}{\lambda p^2}\{\lambda\theta - p^2 - \lambda\xi p\} + \frac{rM^2}{\theta} - \frac{rM\delta_1}{2} - \theta^2\delta_1^2\delta_2 + \frac{r^2}{p^3}\delta_1 - \frac{\theta r\delta_1\delta_3}{2}\right];$$ (4.10b)

$$Q_2 = \frac{1}{\theta^2 D^2}\left[\frac{\theta\,\delta_1 D}{p}(p(\xi-1)-\theta) + \frac{rD}{\lambda p^2}\{\lambda\theta - p^2 - \lambda\xi p\} + \frac{rM\delta_1}{\theta} - \frac{3\delta_1 r\delta_2}{2} + \frac{r^2}{\theta^2 p^3} - \frac{r^2\delta_3}{2\theta}\right];$$ (4.10c)

All the function of right hand side of the eqn.(4.10) are to be evaluated for $\hat{\theta}_{ML}$ and $\hat{p}_{ML}$.

E. Approximate Bayes Estimates under Squared Error Loss function

with equations(4.10)-(4.10c), the different Approximate Bayes estimator Under SELF using Lindley's approximation given by

Special cases:

substituting $\mu(\theta,p) = R = e^{-\theta t^p}$ in eqn.(4.7), we get the Approximate Bayes Estimator of Reliability R=R(t) as

$$\hat{R}_{ABS} = R\left[1 + \frac{Rt^p}{2D}\phi_1 - t^p(Q_1 + \theta\log t\,Q_2)\right]$$; at $(\hat{\theta}_{ML},\hat{p}_{ML})$, (5.1)

Where

$$\phi_1 = Mt^p + \log t(\theta t^p - 1) + \left[2\delta_1 + \frac{rlogt}{\theta}\right]$$

F. Approximate Bayes Estimators under Linex loss function

The Approximate Bayes estimator of a function $\mu = \mu(\theta,p)$ of unknown parameters θ *and* p under LINEX loss function in eqn.(1.6) is given by

$$\hat{\mu}_{ABL} = -\frac{1}{a}log(E_{h^*}(e^{-a\mu}))\ ;$$ (6.1)

where

$$E_{h^*}((e^{-a\mu})|\bar{x}) = \frac{\iint e^{-a\mu} h^*(\theta,p)d\theta\,dp}{\iint h^*(\theta,p)d\theta\,dp};$$ (6.2)

we apply Lindley's Procedure to obtain different Approximate Bayes estimator under LLF as

(i)Approximate Bayes estimator of *R=R(t)* under LLF is

$$\hat{R}_{ABL} = R + \left(-\frac{1}{a}\right)log\phi_3;\ \text{at}(\hat{\theta}_{ML},\hat{p}_{ML}),$$ (6.3)

where

$$\phi_3 = \left[1 + \frac{a\,R\,t^p}{2D}\left\{Mt^p(aR-1) + (\theta t^p(aR-1) + 1)\log t\left(2\delta_1 + \frac{r}{\theta}\,logt\right)\right\} + a\,R\,t^p(Q_1 + \theta\,logt\,Q_2)\right]$$

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