Methods of Eigen Function Expansion to Solve Non-Homogeneous Boundary Value Problems

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Abstract— Among various methods to solve linear boundary value problems defined on finite domain, method of Eigen function expansion is most popular method. In this method, the solution of given boundary value problem is obtained as a series of Eigen functions, that is, as an expansion of Eigen functions. The general theory of Eigen values, Eigen functions and convergence of Eigen function expansions is most important and deepest part of modern mathematics. For this type of expansion, most probably the method of separation of variables is used. To use the method of separation of variables, the partial differential equation must be linear and homogeneous. Nonhomogeneous boundary value problem means, either the problem is nonhomogeneous or the boundary conditions are nonhomogeneous or both. As for example, a nonhomogeneous heat flow problem means either the problem has source term or it has nonzero boundary conditions or it has a source term and nonzero boundary conditions both. For nonhomogeneous partial differential equations, to get Eigen function expansion, using the method of separation of variables, it is required to convert the given nonhomogeneous conditions into homogeneous boundary conditions. For this purpose, it is required to assume an equilibrium function which is a function of variables x or an appropriate reference function, which is a function of variables x and t for one-dimensional problems [1], (for multidimensional problems, a reference function contains spatial variables and time variable t) in such a way that the nonhomogeneous boundary conditions can be converted into homogeneous boundary conditions. So there arises a question: Do we have any method to solve nonhomogeneous problems without converting nonhomogeneous boundary conditions into homogeneous ones? The answer is yes. Eigen function expansion can be done using Green's formula also and in this method, there is no need to convert nonhomogeneous boundary conditions into homogeneous ones. The only drawback is that the solution obtained by this method will not be valid at boundary points and hence convergence of Eigen function expansion, using Green's formula will be slower compared to that of expansion obtained using method of separation of variables[1].In this paper authors want to discuss theory for both the methods: (1) Eigen function expansion using method of separation of variables and (2) Eigen function expansion using Green's formula. To illustrate the methods, one boundary value problem is also solved using both the methods.

Keywords: Method of separation of variables, Eigen Values, Eigen functions, Eigen function expansion, Equilibrium functions, Reference functions, Sturm-Lioville Operator, Green's formula

I. Introduction

It is well-known that a homogeneous, linear boundary value problem can be solved using method of separation of variables.

As for example, the problem, "To solve $\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}$ with $v(0,t) = 0$, $v(L,t) = 0$ and initial condition $v(x,0) = g(x)$", is a linear homogeneous boundary value problem and it can be solved using method of separation of variables. In this method, it is assumed that $v(x,t)$ can be expressed as product of two functions, out of which one is solely of x and another is solely of t.

That is, $v(x,t) = \varphi(x)G(t)$.

Then

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}$$
$$\Rightarrow \varphi(x)G'(t) = k\,\varphi''(x)G(t)$$
$$\Rightarrow \frac{\varphi''(x)}{\varphi(x)} = \frac{G'(t)}{kG(t)} = -\lambda \quad (\lambda > 0\ )$$

(Equality between these two expressions implies that both of them must be a constant.)

Then one gets two ordinary differential equations:

$\frac{d^2\varphi}{dx^2} + \lambda\varphi = 0$ with $\varphi(0) = 0$ and $\varphi(L) = 0$ as well as $\frac{dG}{dt} + \lambda kG = 0$

Solving upon,

$\frac{d^2\varphi}{dx^2} + \lambda\varphi = 0$, one gets $\varphi(x) = c_1\cos\sqrt{\lambda}\,x + c_2\sin\sqrt{\lambda}\,x$.

Applying boundary conditions, one gets:

$\varphi(x) = \sin\sqrt{\lambda}\,x = \sin\left(\frac{n\pi x}{L}\right)$.

The separation constant λ is known as an *Eigen value.* Here $\lambda = \left(\frac{n\pi}{L}\right)^2, n \in N$

A nontrivial solution $\varphi(x)$, existing only for certain Eigen value, is known as an Eigen function, corresponding to that particular Eigen value λ.

Here $\varphi(x) = \sin\left(\frac{n\pi x}{L}\right)$ is Eigen function corresponding to the Eigen value $\lambda = \left(\frac{n\pi}{L}\right)^2$.

Solving $\frac{dG}{dt} + \lambda kG = 0$, one gets

$$G(t) = c_3\,e^{-\lambda kt} = c_3 e^{-\frac{(n\pi)^2 kt}{L^2}}.$$

So the solution is: $v(x,t) = c_2c_3\sin\left(\frac{n\pi x}{L}\right)\,e^{-\frac{(n\pi)^2 kt}{L^2}}$.

Applying principle of superposition, the solution is:

$v(x,t) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)\,e^{-\frac{(n\pi)^2 kt}{L^2}}$,

where b_n's are Fourier coefficients, defined as

$$b_n = \frac{2}{L}\int_0^L v(x,0)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Thus solution is obtained as an expansion of Eigen functions.

Suppose, the given problem has nonhomogeneous boundary conditions. If boundary conditions are nonhomogeneous, but time-independent, then an equilibrium function $u_E(x)$ is to be assumed. If boundary conditions, are nonhomogeneous, but time dependent, a reference temperature distribution $r(x,t)$ is to be assumed. Whether it is an equilibrium function $u_E(x)$ or a reference function $r(x,t)$, both should satisfy the same boundary conditions as given for the unknown solution $u(x,t)$ in the problem.

As for example, if problem is like:

Solve $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + k$, with $u(0,t) = A$, $u(L,t) = B$,

where A and B are constants, then boundary conditions are time-independent, so an equilibrium function $u_E(x)$ is to be assumed and it should satisfy $u_E(0) = u(0,t) = A$ and $u_E(L) = u(L,t) = B$.

The substitution $v(x,t) = u(x,t) - u_E(x)$, implies,

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + k = k\left(\frac{\partial^2 v}{\partial x^2} + \frac{d^2 u_E}{dx^2}\right) + k.$$

So $u_E(x)$ must satisfy $k\frac{d^2 u_E}{dx^2} + k = 0$.

This implies $\frac{d^2 u_E}{dx^2} = -1 \Rightarrow u_E(x) = -\frac{x^2}{2} + \left(\frac{B-A}{L} + \frac{L}{2}\right)x + A$

Hence $v(x,t)$ satisfies $\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}$ with $v(0,t) = 0$ and $v(L,t) = 0$

Then one can apply the method above discussed to obtain $v(x,t)$, and hence can find $u(x,t)$.

To solve a problem with time-dependent boundary conditions, say $u(0,t) = A(t)$, $u(L,t) = B(t)$, where $A(t)$ and $B(t)$ are functions of t, a reference function is to be assumed.

Here relative reference function $r(x,t)$ will be

$$r(x,t) = A(t) + \left(\frac{B(t)-A(t)}{L}\right)x.$$

The function $v(x,t) = u(x,t) - r(x,t)$, then satisfies homogeneous boundary conditions, but now as

$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t} - \frac{\partial r}{\partial t} = \frac{\partial u}{\partial t} - A'(t) - \left(\frac{B'(t)-A'(t)}{L}\right)x$, there would be some contribution in source term. Hence given problem would be a nonhomogeneous problem with homogeneous boundary conditions.

So, in all, the main concern is to be able to solve a nonhomogeneous problem with homogeneous boundary conditions, like:

Problem-1: Solve $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$ (1)

with $u(0,t) = 0$, $u(L,t) = 0$ (1-a,b)

and $u(x,0) = f(x)$ (1-c)

In this paper authors intend to discuss two methods to solve such problem, the method of Eigen function expansion using separation of variables and the method of Eigen function expansion using Green's formula.

II. METHODS

A. *The method of Eigen function expansion using separation of variables:*

Suppose $v(x,t)$ satisfies the related homogeneous problem to problem-1. That is:

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} \quad \text{with} \quad v(0,t) = 0\,,\, v(L,t) = 0 \qquad (2)$$

The Eigen functions $\emptyset_n$ of this related homogeneous problem will satisfy:

$$\frac{d^2\varphi_n}{dx^2} + \lambda_n\varphi_n = 0$$

with boundary conditions, $\varphi_n(0) = 0$, $\varphi_n(L) = 0$.

Here Eigen functions will be

$\varphi_n(x) = \sin\frac{n\pi x}{L}$, corresponding to Eigen values $\lambda_n = (n\pi/L)^2$ $n \in N$ and the solution of (2), is

$$v(x,t) = \sum_{n=1}^{\infty} a_n\varphi_n(x)$$

where a_n's are Fourier coefficients.

To develop method of Eigen function expansion for unknown function $u(x,t)$, in (1), it is assumed that $u(x,t)$ can be expanded as a series of the related homogenous Eigen functions, but the coefficients now, will be functions of variable t instead of being merely constants.

(as Eigen functions satisfy $\frac{d^2\varphi_n}{dx^2} + \lambda_n\varphi_n = 0$, this assumption is quite valid.)

That is, $u(x,t)$ can be expressed as generalized Fourier series, as:

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\varphi_n(x) \qquad (3)$$

where the generalized Fourier coefficients $a_n(t)'s$ are functions of t.

As

$$u(x,0) = \sum_{n=1}^{\infty} a_n(0)\varphi_n(x) = f(x),\ a_n(0) = \frac{\int_0^L f(x)\varphi_n(x)dx}{\int_0^L \varphi_n{}^2(x)dx} \qquad (4)$$

(using orthogonality of the Eigen functions)

Hence $\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\frac{da_n(t)}{dt}\varphi_n(x)$ and

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} a_n(t)\frac{d^2\varphi_n}{dx^2} = -\sum_{n=1}^{\infty} a_n(t)\lambda_n\varphi_n$$

(Continuity of $u(x,t)$ and $\frac{\partial u}{\partial x}$ allow the term by term differentiation and as $\frac{d^2\varphi_n}{dx^2} + \lambda_n\varphi_n = 0$)

So $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{da_n(t)}{dt}\varphi_n(x) = -\sum_{n=1}^{\infty} ka_n(t)\lambda_n\varphi_n + Q(x,t)$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(\frac{da_n(t)}{dt} + ka_n(t)\lambda_n\right)\varphi_n(x) = Q(x,t) \qquad (5)$$

As (5) represents the generalized Fourier series of $Q(x,t)$, it is clear that

$$\left(\frac{da_n(t)}{dt} + ka_n(t)\lambda_n\right) = \frac{\int_0^L Q(x,t)\varphi_n(x)dx}{\int_0^L \varphi_n{}^2(x)dx} = q_n(t) \quad \text{(say)}$$

Thus $\frac{da_n(t)}{dt} + ka_n(t)\lambda_n = q_n(t)$ (6)

Equation (6) is a first order linear differential equation in $a_n(t)$.

One can solve as:

$$e^{k\lambda_n t}\left(\frac{da_n(t)}{dt} + ka_n(t)\lambda_n\right) = e^{k\lambda_n t}q_n(t)$$

$$\Rightarrow \frac{d}{dt}\left(e^{k\lambda_n t}a_n(t)\right) = e^{k\lambda_n t}q_n(t) \quad \Rightarrow \left[e^{k\lambda_n t}a_n(t)\right]_0^t = \int_0^t e^{k\lambda_n\tau}q_n(\tau)\,d\tau$$

$$\Rightarrow e^{k\lambda_n t}a_n(t) - a_n(0) = \int_0^t e^{k\lambda_n\tau}q_n(\tau)\,d\tau$$

$$\Rightarrow a_n(t) = e^{-k\lambda_n t}a_n(0) + e^{-k\lambda_n t}\int_0^t e^{k\lambda_n\tau}q_n(\tau)\,d\tau$$

Substituting $a_n(t)$ in (3), the solution will be

$$u(x,t) = \sum_{n=1}^{\infty}\left(e^{-k\lambda_n t}a_n(0)\, e^{-k\lambda_n t}\int_0^t e^{k\lambda_n \tau} q_n(\tau)\,d\tau\right)\varphi_n(x) \quad (7)$$

Where $a_n(0) = \frac{\int_0^L f(x)\varphi_n(x)dx}{\int_0^L \varphi_n{}^2(x)dx}$ and $q_n(t) = \frac{\int_0^L Q(x,t)\varphi_n(x)dx}{\int_0^L \varphi_n{}^2(x)dx}$. (8)

If $Q(x,t) = 0$, that is, if the problem is homogeneous, then in that case, $q_n(t) = 0$ and the solution will be $u(x,t) = \sum_{n=1}^{\infty}\left(e^{-k\lambda_n t}a_n(0)\right)\emptyset_n(x)$, as expected.

As for this problem, $\varphi_n(x) = \sin\frac{n\pi x}{L}$, corresponding to Eigen values $\lambda_n = (n\pi/L)^2$, using (8),

$a_n(0) = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$ and

$q_n(t) = \frac{2}{L}\int_0^L Q(x,t)\sin\frac{n\pi x}{L}dx.$

Substituting these values in (7) the solution will be:

$$u(x,t) = \sum_{n=1}^{\infty}\left(e^{-k\left(\frac{n\pi}{L}\right)^2 t}\left(\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx\right) + e^{-k\left(\frac{n\pi}{L}\right)^2 t}\int_0^t e^{k\left(\frac{n\pi}{L}\right)^2\tau}\left(\frac{2}{L}\int_0^L Q(x,\tau)\varphi_n(x)dx\right)d\tau\right)\sin\frac{n\pi x}{L} \quad (9)$$

B. Method of Eigen function expansion using Green's Formula:

Eigen function expansion can be done using Green's formula also. For that purpose, firstly Green's formula is introduced:

1) Green's formula:

Let L denote linear differential operator $L \equiv \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x)$, where $p(x)$ and $q(x)$ are continuous functions with $p(x) > 0$ everywhere in the entire domain. This operator is known as Sturm-Liouville operator. Using it, a regular Sturm-Liouville Eigen value problem,

$\frac{d}{dx}\left[p(x)\frac{d\varphi}{dx}\right] + q(x)\varphi + \lambda\sigma(x)\varphi = 0, \quad a \le x \le b$ is represented as $L(\varphi) + \lambda\sigma(x)\varphi = 0.$

Let u and v be any two functions (not necessarily Eigen functions), then

$$uL(v) - vL(u) = u\left(\frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x)\right)v - v\left(\frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x)\right)u = \frac{d}{dx}\left[p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right]$$

$$\Rightarrow \int_a^b (uL(v) - vL(u))dx = \left[p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right]_a^b \quad (10)$$

Result (10) is known as Green's formula for one-dimensional Eigen value problems. It is a powerful analytic tool to study second order linear partial differential equations[3].

To illustrate the method, considering the same Problem-1:

$$\text{Solve } \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t) \quad (11)$$

$$u(0,t) = A(t) \text{ and } u(L,t) = B(t) \quad (11\text{-a,b})$$

$$u(x,0) = f(x) \quad (11\text{-c})$$

The Eigen functions $\varphi_n(x)$ of the related homogeneous problem satisfy

$\frac{d^2\varphi_n}{dx^2} + \lambda_n\varphi_n = 0$ with $\varphi_n(0) = 0$ and $\varphi_n(L) = 0$.

Here $\varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ are Eigen functions, corresponding to the Eigen value $\lambda_n = \left(\frac{n\pi x}{L}\right)^2$ Here u satisfies nonhomogeneous boundary conditions, $u(0,t) = A(t)$ and $u(L,t) = B(t)$, even then also, as u is piecewise smooth function, it can be expressed as expansion of Eigen functions [1].

That is: $u(x,t) = \sum_{n=1}^{\infty} b_n(t)\varphi_n(x)$ (12)

The expression (12) cannot be valid at $x = 0$ and at $x = L$ because $u(x,t)$ satisfies non-homogeneous boundary conditions, whereas $\varphi_n(x)$ satisfies homogeneous boundary conditions. So term by term differentiation with respect to x is not at all justified but the term by term differentiation with respect to t is justified.

Hence,

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\varphi_n(x) \Rightarrow \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\frac{db_n(t)}{dt}\varphi_n(x) \quad (13)$$

Using (11) in (13) one gets: $k\frac{\partial^2 u}{\partial x^2} + Q(x,t) = \sum_{n=1}^{\infty}\frac{db_n(t)}{dt}\varphi_n(x)$ Using the orthogonality of Eigen functions,

$$\frac{db_n(t)}{dt} = \frac{\int_0^L\left[k\frac{\partial^2 u}{\partial x^2} + Q(x,t)\right]\varphi_n(x)dx}{\int_0^L \varphi_n{}^2(x)dx} \quad (14)$$

Let generalized Fourier series of $Q(x,t)$ be $Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\varphi_n(x)$,

then $q_n(t) = \frac{\int_0^L Q(x,t)\varphi_n(x)dx}{\int_0^L \varphi_n{}^2(x)dx}$ (15)

Substituting (15) in (14), one gets:

$$\frac{db_n(t)}{dt} = q_n(t) + \frac{\int_0^L k\frac{\partial^2 u}{\partial x^2}\varphi_n(x)dx}{\int_0^L \varphi_n{}^2(x)dx} \quad (16)$$

To evaluate the integral $\int_0^L k\frac{\partial^2 u}{\partial x^2}\varphi_n(x)dx$ in (14), Green's formula can be utilized.

According to Green's formula,

$$\int_a^b (uL(v) - vL(u))dx = \left[p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right]_a^b$$

Here, for this problem, $L \equiv \frac{\partial^2}{\partial x^2}$ and the interval is $0 \le x \le L$.

Comparing with Sturm-Liouville operator $L \equiv \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x)$, here $p = 1$, $q = 0$.

Also taking $v = \varphi_n$, Green's formula implies that,

$$\int_0^L\left[u\frac{\partial^2\varphi_n}{\partial x^2} - \varphi_n\frac{\partial^2 u}{\partial x^2}\right]dx = \left(u\frac{d\varphi_n}{dx} - \varphi_n\frac{du}{dx}\right)\Big|_0^L \quad (17)$$

Also, $\varphi_n(0) = 0$, $\varphi_n(L) = 0$ and $\frac{d^2\varphi_n}{dx^2} = -\lambda_n\varphi_n$.

Substituting these results in (17), one gets

$$\int_0^L\left(u(-\lambda_n\varphi_n) - \varphi_n\frac{\partial^2 u}{\partial x^2}\right)dx = u(L,t)\left(\frac{d\varphi_n}{dx}\right)_{x=L} - u(0,t)\left(\frac{d\varphi_n}{dx}\right)_{x=0}$$

$$\Rightarrow \int_0^L \varphi_n\frac{\partial^2 u}{\partial x^2}dx = -\lambda_n\int_0^L u\varphi_n(x)dx - \left[B(t)\left(\frac{d\varphi_n}{dx}\right)_{x=L} - A(t)\left(\frac{d\varphi_n}{dx}\right)_{x=0}\right] \quad (18)$$

As $b_n(t)$'s are generalized Fourier coefficients of $u(x,t)$,

$$b_n(t) = \frac{\int_0^L u\varphi_n(x)dx}{\int_0^L \varphi_n{}^2(x)dx} \quad (19)$$

Using (19) and (18) in (16), one gets a first order linear differential equation:

$$\frac{db_n(t)}{dt} + k\lambda_n b_n = q_n(t) + \frac{k\left[A(t)\left(\frac{d\varphi_n}{dx}\right)_{x=0} - B(t)\left(\frac{d\varphi_n}{dx}\right)_{x=L}\right]}{\int_0^L \varphi_n{}^2(x)dx} \quad (20)$$

Also,

$$\varphi_n(x) = \sin\left({}^{n\pi x}/_L\right)$$

$$\Rightarrow \int_0^L \varphi_n{}^2(x)dx = \int_0^L \left(\sin\left({}^{n\pi x}/_L\right)\right)^2 dx = \frac{L}{2}$$

and $\frac{d\varphi_n}{dx} = \frac{n\pi}{L}\cos\frac{n\pi x}{L} \Rightarrow \left(\frac{d\varphi_n}{dx}\right)_{x=0} = \frac{n\pi}{L}$ as well as $\left(\frac{d\varphi_n}{dx}\right)_{x=L} = \frac{n\pi}{L}(-1)^n$

Hence now, (20) becomes:

$$\frac{db_n(t)}{dt} + k\lambda_n b_n = q_n(t) + \frac{k\left(\frac{n\pi}{L}\right)[A(t) - (-1)^n B(t)]}{{}^L/_2}$$

$$\Rightarrow \frac{db_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 b_n = q_n(t) + \frac{2kn\pi[A(t) - (-1)^n B(t)]}{L^2} = \tilde{q}_n(t)$$

$$\Rightarrow \frac{db_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 b_n = \tilde{q}_n(t) \quad (21)$$

By solving this first order differential equation (21), one can find out $b_n(t)$ and hence can find $u(x,t)$ using (12).Thus using Green's formula, the problem can be solved without changing nonhomogeneous boundary conditions into homogeneous ones.

III. SOLUTIONS

One such problem is solved here using both the methods to illustrate.

Problem-2

Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sin 5x\, e^{-2t}$, subject to $u(x,0) = 0$, $u(0,t) = 1$ and $u(\pi,t) = 0$

A. *Solution using the method of Eigen function expansion using separation of variables:*

Here the given equation is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sin 5x\, e^{-2t} \quad (22)$$

With $u(x,0) = 0$, $u(0,t) = 1$, $u(\pi,t) = 0$ (22-a,b,c)

Here $u(0,t) = 1$, $u(\pi,t) = 0$, so, here equilibrium function will be, $T(x) = 1 - \frac{x}{\pi}$

Let $v(x,t) = u(x,t) - T(x) \Rightarrow u(x,t) = v(x,t) + T(x)$

So (22) implies ,

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \sin 5x\, e^{-2t} \quad (23)$$

With $v(x,0) = u(x,0) - T(x) = \frac{x}{\pi} - 1$ and $v(0,t) = 0$, $v(\pi,t) = 0$ (23-a,b,c)

Thus v satisfies non-homogeneous equation with homogeneous boundary conditions.

Using Eigen function expansion method, by separation of variables, the Eigen functions will be $\emptyset_n(x) = \sin nx$ corresponding to Eigen values $\lambda_n = n^2$, $n \in N$.

So let

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\emptyset_n(x) = \sum_{n=1}^{\infty} a_n(t)\sin nx \quad (24)$$

be the solution of (22).

Then

$$v(x,0) = \sum_{n=1}^{\infty} a_n(0)\sin nx$$

$$\Rightarrow \frac{x}{\pi} - 1 = \sum_{n=1}^{\infty} a_n(0)\sin nx$$

So,

$$a_n(0) = \frac{2}{\pi}\int_0^{\pi}\left(\frac{x}{\pi} - 1\right)\sin nx\, dx = \frac{2}{\pi}\left[\left(\frac{x}{\pi} - 1\right)\left(\frac{-\cos nx}{n}\right) + \frac{1}{\pi}\cdot\frac{\sin nx}{n^2}\right]_0^{\pi} = -\frac{2}{n\pi} \quad (25)$$

Equation (22) implies

$$\frac{\partial}{\partial t}\left(\sum_{n=1}^{\infty} a_n(t)\sin nx\right) = \frac{\partial^2}{\partial x^2}\left(\sum_{n=1}^{\infty} a_n(t)\sin nx\right) + \sin 5x\, e^{-2t}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{da_n(t)}{dt}\sin nx = -\sum_{n=1}^{\infty} a_n(t)n^2\sin nx + \sin 5x\, e^{-2t}$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(\frac{da_n}{dt} + a_n(t)n^2\right)\sin nx = \sin 5x\, e^{-2t}$$

This implies $\frac{da_n(t)}{dt} + a_n(t)n^2 = \begin{cases} 0, & n \neq 5 \\ e^{-2t}, & n = 5 \end{cases}$

Multiplying by integrating factor $e^{n^2 t}$ both sides we have

$$e^{n^2 t}\left(\frac{da_n(t)}{dt} + a_n(t)n^2\right) = \begin{cases} 0, & n \neq 5 \\ e^{(n^2-2)t}, & n = 5 \end{cases}$$

$$\Rightarrow \frac{d}{dt}\left(e^{n^2 t}a_n(t)\right) = \begin{cases} 0, & n \neq 5 \\ e^{(n^2-2)t} & n = 5 \end{cases}$$

$$\Rightarrow \left[e^{n^2 t}a_n(t)\right]_0^t = \begin{cases} 0, & n \neq 5 \\ \frac{e^{(n^2-2)t} - 1}{(n^2-2)}, & n = 5 \end{cases}$$

$$\Rightarrow e^{n^2 t}a_n(t) - a_n(0) = \begin{cases} 0, & n \neq 5 \\ \dfrac{e^{(n^2-2)t} - 1}{(n^2-2)}, & n = 5 \end{cases}$$

$$\Rightarrow a_n(t) = e^{-n^2 t}\left(a_n(0)\right) + \begin{cases} 0, & n \neq 5 \\ \left(e^{-n^2 t}\right)\left(\dfrac{e^{(n^2-2)t} - 1}{(n^2-2)}\right), & n = 5 \end{cases}$$

$$\Rightarrow a_n(t) = -\frac{2}{n\pi}e^{-n^2 t} + \begin{cases} 0, & n \neq 5 \\ \dfrac{e^{-2t} - e^{-n^2 t}}{(n^2-2)}, & n = 5 \end{cases}$$

Hence solution of (22) is,

$$v(x,t) = \sum_{\substack{n=1,\\ n\neq 5}}^{\infty} a_n(t)\sin nx = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{e^{-n^2 t}}{n}\sin nx + \left\{-\frac{2}{5\pi}e^{-25t} + \frac{e^{-2t} - e^{-25t}}{23}\right\}\sin 5x$$

So

$$u(x,t) = -\frac{2}{\pi}\sum_{\substack{n=1,\\ n\neq 5}}^{\infty}\frac{e^{-n^2 t}}{n}\sin nx + \left\{-\frac{2}{5\pi}e^{-25t} + \frac{e^{-2t} - e^{-25t}}{23}\right\}\sin 5x + \left(1 - \frac{x}{\pi}\right)$$

Hence

$$u(x,t) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{e^{-n^2t}}{n}\sin nx + \left\{\frac{e^{-2t}-e^{-25t}}{23}\right\}sin5x + \left(1-\frac{x}{\pi}\right) \quad (26)$$

Result (26) represents solution of problem-2.

B. Solution by Method of Eigen function expansion using Green's Formula:

Solving the same problem,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + sin\ 5x\ e^{-2t}, \quad (27)$$

subject to $u(0,t) = 1$ and $u(\pi,t) = 0$ and $u(x,0) = 0$ (27-a,b,c) without reducing to homogeneous boundary conditions.

The problem is to be solved without reducing nonhomogeneous boundary conditions to homogeneous boundary conditions. So the method of Eigen function expansion by Green's formula, will be used.

Comparing expressions in (27) and (27-a,b,c) with (11) and (11-a,b,c) respectively,

one gets: $Q(x,t) = \sin 5x e^{-2t}, k = 1$ and $L = \pi$ as well as $u(0,t) = A(t) = 1,\ u(\pi,t) = B(t) = 0,\ u(x,0) = f(x) = 0$

Here Eigen function will be $\varphi_n(x) = \sin(nx)$ corresponding to Eigen value $\lambda_n = n^2,\ n \in N$

Let $u(x,t) = \sum_{n=1}^{\infty} b_n(t)\varphi_n(x)$ be the solution of given problem.

Hence $u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin nx$ (28)

According to result (26),

$$\frac{db_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 b_n = \tilde{q}_n(t) = q_n(t) + \frac{2kn\pi[A(t) - (-1)^n B(t)]}{L^2}$$

So here

$$\frac{db_n}{dt} + n^2 b_n = q_n(t) + \frac{2n(1-0)}{\pi} = q_n(t) + \frac{2n}{\pi} \quad (29)$$

where

$$q_n(t) = \frac{\int_0^L Q(x,t)\varphi_n(x)dx}{\int_0^L \varphi_n{}^2(x)dx} = \frac{\int_0^\pi (\sin 5x\ \ e^{-2t})(\sin nx\ )dx}{\int_0^\pi sin^2(nx)dx}$$

$$\Rightarrow q_n(t) = \frac{e^{-2t}.\frac{1}{2}\int_0^\pi (\cos(n-5)x - \cos(n+5)x\ )\ dx}{\frac{\pi}{2}}$$

$$\Rightarrow q_n(t) = \begin{cases} \dfrac{e^{-2t}\int_0^\pi (\ 1-\cos(\ 10x\ )\ dx}{\pi} & if\ n = 5 \\ \dfrac{e^{-2t}\int_0^\pi (\ \cos(n-5)x - \cos(\ n+5)x\ )\ dx}{\pi} & if\ n \neq 5 \end{cases}$$

$$\Rightarrow q_n(t) = \begin{cases} e^{-2t} & if\ n = 5 \\ 0 & if\ n \neq 5 \end{cases}$$

Substituting this value in (29), one gets:

$$\frac{db_n}{dt} + n^2 b_n = \frac{2n}{\pi} + \begin{cases} e^{-2t} & if\ n = 5 \\ 0 & if\ n \neq 5 \end{cases}$$

Solving this differential equation one gets:

$$\frac{d}{dt}\left(\ e^{n^2t}b_n(t)\right) = \begin{cases} e^{n^2t}\left(\frac{2n}{\pi} + e^{-2t}\ \right) & if\ n = 5 \\ e^{n^2t}\frac{2n}{\pi} & if\ n \neq 5 \end{cases}$$

$$\Rightarrow \left[e^{n^2t}\ b_n(t)\right]_0^t = \begin{cases} \int_0^t \left(e^{n^2t}\left(\frac{2n}{\pi}\right) + e^{(n^2-2)t}\right)dt & if\ n = 5 \\ \int_0^t \left(e^{n^2t}\left(\frac{2n}{\pi}\right)\right)dt & if\ n \neq 5 \end{cases}$$

$$\Rightarrow e^{n^2t}\ b_n(t) - b_n(0) = \begin{cases} \int_0^t \left(e^{25t}\left(\frac{10}{\pi}\right) + e^{23t}\right)dt & if\ n = 5 \\ \int_0^t e^{n^2t}\left(\frac{2n}{\pi}\right)dt & if\ n \neq 5 \end{cases}$$

$$\Rightarrow e^{n^2t}\ b_n(t) - b_n(0) = \begin{cases} \left[e^{25t}\left(\frac{2}{5\pi}\right) + \frac{e^{23t}}{23}\right]_0^t & if\ n = 5 \\ \left[e^{n^2t}\left(\frac{2}{\pi n}\right)\right]_0^t & if\ n \neq 5 \end{cases}$$

$$\Rightarrow e^{n^2t}\ b_n(t) - b_n(0) = \begin{cases} (e^{25t}-1)\left(\frac{2}{5\pi}\right) + \frac{(e^{23t}-1)}{23} & if\ n = 5 \\ (e^{n^2t}-1)\left(\frac{2}{n\pi}\right) & if\ n \neq 5 \end{cases} \quad (30)$$

Now (28) and (27-c) imply that $u(x,0) = \sum_{n=1}^{\infty} b_n(0)\sin nx\ = 0$

This implies $b_n(0) = 0$. Substituting this value in (30), one gets

$$e^{n^2t}\ b_n(t) = \begin{cases} (e^{25t}-1)\left(\frac{2}{5\pi}\right) + \frac{(e^{23t}-1)}{23} & if\ n = 5 \\ (e^{n^2t}-1)\left(\frac{2}{n\pi}\right) & if\ n \neq 5 \end{cases}$$

$$\Rightarrow b_n(t) = \begin{cases} e^{-25t}\left\{(e^{25t}-1)\left(\frac{2}{5\pi}\right) + \frac{(e^{23t}-1)}{23}\right\} & if\ n = 5 \\ e^{-n^2t}\left\{(e^{n^2t}-1)\left(\frac{2}{n\pi}\right)\right\} & if\ n \neq 5 \end{cases}$$

$$\Rightarrow b_n(t) = \begin{cases} (1-e^{-25t})\left(\frac{2}{5\pi}\right) + \frac{(e^{-2t}-e^{-25t})}{23} & if\ n = 5 \\ \left(1-e^{-n^2t}\right)\left(\frac{2}{n\pi}\right) & if\ n \neq 5 \end{cases} \quad (31)$$

Using (31) in (28), one gets

$$u(x,t) = \sum_{\substack{n=1 \\ n\neq 5}}^{\infty}\left(1-e^{-n^2t}\right)\left(\frac{2}{n\pi}\right)\sin nx + \left((1-e^{-25t})\left(\frac{2}{5\pi}\right) + \frac{(e^{-2t}-e^{-25t})}{23}\right)\sin 5x$$

Hence

$$u(x,t) = \sum_{n=1}^{\infty}\left(1-e^{-n^2t}\right)\left(\frac{2}{n\pi}\right)\sin nx + \frac{(e^{-2t}-e^{-25t})}{23}\sin 5x \quad (32)$$

It can be observed that $u(x,t)$ in (32) does not satisfy boundary conditions, but otherwise it represents the solution of the problem.

IV. CONCLUSIONS

For any function $\xi(x,t)$, if it is expressed as series of Eigen functions $\varphi_n(x)$, as $\xi(x,t) = \sum_{n=1}^{\infty} \beta_n(t)\varphi_n(x)$,

1) the series will converge comparatively fast, if $\xi(x,t)$ and $\varphi_n(x)$ satisfy same homogeneous boundary conditions.
2) If $\xi(x,t)$ does not satisfy homogeneous boundary conditions as those are satisfied by Eigen functions $\varphi_n(x)$, then convergence of the series will be slower everywhere. Also, in this case $\sum_{n=1}^{\infty} \beta_n(t)\varphi_n(x)$ will not satisfy nonhomogeneous boundary conditions, which are demanded for $\xi(x,t)$ in the problem.
3) One should prefer to convert nonhomogeneous boundary conditions into homogeneous boundary conditions to establish faster convergence. If it becomes comparatively difficult or tedious, to determine a reference function, then method of Eigen function expansion using Green's formula can be used.

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