

Applications of the Fourier-Wiener and Integral Transforms

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Abstract— We shall study the existence and uniqueness and regularity of solutions for Cauchy problems associated with the following two equations: $v_t = -R^k v$, $v_{tt} = -R^k v$; and the elliptic type equation $-R^k v = f$ ($k > 1$). We applications of the Fourier-Wiener transform defined on a class of polynomial cylinder functions on Hilbert space and Wiener space.

Keywords: Wiener Space, Wiener Measure, Hilbert Space, Exponential Type Analytic Functions, Equicontinuous Semigroup of Class (C_0) .

I. INTRODUCTION

We study the Fourier-Wiener transform was defined on a class of mean exponential type analytic functionals on the classical Wiener space \mathcal{C} and applications of the Fourier-Wiener transform are concerned we find that it is desirable to enlarge the domain of definition of the Fourier-Wiener transform as much as possible and define on the class of exponential type analytic functions \mathcal{E}_α on $[B]$ - the complexification of B .

Let \mathcal{E}_α be the class of exponential type analytic functions defined on the complexification $[B]$ of B . For each pair of nonzero complex numbers α, β and $f \in \mathcal{E}_\alpha$, we define

$$F_\alpha f(y) = \int f(\alpha x + \beta y) p_1(dx)(y \in [B]).$$

We show that the inverse $F_{\alpha, \beta}^{-1}$ exists and there exist two nonzero complex numbers α', β' such that $F_{\alpha, \beta}^{-1} = F_{\alpha', \beta'}$ where α', β' is a solution of $\beta\beta' = 1$ and $(\beta\alpha')^2 + \alpha^2 = 0$.

II. PRELIMINARIES

We shall apply the Fourier-Wiener transform [99] and integration by parts [100] to examine Cauchy problems associated with the following types of differential equations:

- 1) $(\cdot), v_t(x, t) = -R^k v(x, t)$
- 2) $v_t(x, t) = -R^k v(x, t)$, and also the elliptic type differential equation (iii) $-R^k v(x) = g(x)$ where $k \geq 1$.

Theorem (1.2): For $f \in \mathcal{E}_\alpha$, we have

$$F_c^{-1} F_c(z) = f(z), \quad z \in [B]. \quad (1.2)$$

Proof: (i) We claim that (1) holds for $(z) = Tz^m, m = 1, 2, \dots$, with T a symmetric operator in $[W_m]$. In fact,

$$\begin{aligned} F_c^{-1} F_c(T(\cdot)^m)(z) &= \int \int_{B, B} T(x + iy + z)^m p_c(dx) p_c(dy) \\ &= Tz^m + \int \int_B \sum_{j=0}^{m-1} Tz^j (x + iy)^{m-j} p_c(dx) p_c(dy) \end{aligned}$$

Let $\int \int_B \sum_{j=0}^{m-1} Tz^j (x + iy)^{m-j} p_c(dx) p_c(dy) = 0$, by Lemma (4.1.12)[93]

$$= Tz^m$$

3) By Proposition (4.1.8)[93], we have

$$f(z) = \sum_{m=0}^{\infty} (1/m!) D^m f(0) z^m.$$

Consequently,

$$F_c f(z) = \sum_{m=0}^{\infty} (1/m!) F_c(D^m f(0)(\cdot)^m)(z),$$

by Lebesgue's dominated convergence theorem.

Finally, by Step (i) and the dominated convergence theorem again,

$$\begin{aligned} F_c^{-1} F_c f(z) &= \sum_{m=0}^{\infty} (1/m!) F_c^{-1} F_c(D^m f(0)(\cdot)^m)(z) \\ &= \sum_{m=0}^{\infty} (1/m!) D^m f(0) z^m = f(z). \end{aligned}$$

Theorem (2.2): There is a unique solution $u(y, t)$ for the Cauchy problem $u_t(y, t) = -R^k u(y, t); u(y, 0) = F_1 f(y) (y \in B, t > 0)$ such that (i) $u(\cdot, t) \in \tilde{\mathcal{E}}_\alpha$ for each $t \geq 0$, (ii) $u(y, t)$ is (strongly) differentiable in t (i.e. $(\partial/\partial t)u(y, t)$ exists uniformly on bounded sets). Moreover, the solution is given by $(y, t) = T_t(F_1 f)(y)$.

Proof: Let $g(y, s)$ be any solution for $u_t(y, t) = -R^k u(y, t); u(y, 0) = F_1 f(y) (y \in B, t > 0)$ such that (i) and (ii) hold.

Fix $t > 0$. Set $(y, s) = T_{t-s} g(y, s)$ for $t \geq s \geq 0, y \in [B]$. Since $g(\cdot, s) \in \tilde{\mathcal{E}}_\alpha$, the domain of $-R^k$, it follows by [112, Chapter 9, Theorem 2] that $F(y, s)$ is strongly differentiable and $(\partial/\partial s)F(y, s) = -R^k T_{t-s} g(y, s) + T_{t-s}(\partial/\partial s)g(y, s)$ (since T_t is an equicontinuous semigroup and g satisfies (ii))

$$\begin{aligned} &= T_{t-s} (-R^k g(y, s) - (\partial/\partial s)g(y, s)) \\ &= 0, \quad y - \text{uniformly on bounded sets.} \end{aligned}$$

This implies that $(y, s) = C$ (a constant) on $[B]$ for $0 \leq s \leq t$. In particular, $(y, t) = F(y, 0) = T_t g(y, 0) = T_0 g(y, 0) = g(y, t)$. We conclude that $g(y, t) = T_t(F_1 f)(y)$.

Proposition (2.3): $\{T_t\}$ is an equicontinuous semigroup of class (C_0) [112, Chapter 9].

Proof: Obviously,

$$T_{t+s} = T_t T_s \text{ and } T_0 = I. \quad (2.2)$$

By the mean value theorem, we have (for $t_0 \geq 0$)

$$\|T_t f - T_{t_0} f\| \leq \sum_{n=0}^{\infty} (1/n!) \|e^{-m^* t} - e^{-m^* t_0}\| \|D^n f(0) x^n\|$$

$$\leq |t - t_0| \|f\|_{2M} \left(\sum_{n=1}^{\infty} (n^k/2^m) \right).$$

By $(1/m!) |D^m(0)x^m| \leq 2^{-m} \|f\|_{2M}$ (3.2)

Thus

$$\lim_{t \rightarrow t_0} T_t f = T_{t_0} f \quad \text{for } t_0 \geq 0. \quad (4.2)$$

Finally, suppose q is any continuous seminorm on $\tilde{\mathcal{E}}_a$. Then there exist N and a constant C_1 such that $(f) \leq C_1 \|f\|_N$.

Consequently, $(T_t f) \leq C_1 \|T_t f\|_N \leq 2C_1 \|f\|_{2N}$, i.e. $\{T_t\}$ is equicontinuous.

III. CLAIMS

A. *Theorem (1.3):*

- 1) In order that equation $-R^k(x) = f(x)$ has a solution in \mathcal{E}_a , it is necessary and sufficient that $f \in \mathcal{E}_a$ and $\mathcal{F}_1 f(0) = 0$.
- 2) The solution of $-R^k(x) = f(x)$ is unique in the sense that if $W_1(x), W_2(x) \in \mathcal{E}_a$ and each solves $-R^k u(x) = f(x)$ then $W_1 - W_2$ is a constant.

B. *Proof:*

- 1) Necessity. Since $(\mathcal{E}_a) \subset \mathcal{E}_a$ (Lemma (4.1.17))[93], if $u(x)$ is a solution in \mathcal{E}_a then $R^k u(x) \in \mathcal{E}_a$ so does f . Next, applying the Fourier-Wiener transform to $-R^k u(x) = f(x)$, we get $-\tilde{R}^k(\mathcal{F}_1 u)(y) = \mathcal{F}_1 f(y)$. (1.3)

Therefore $\mathcal{F}_1 f(0) = -\tilde{R}^k(\mathcal{F}_1 u)(0) = 0$.

- 2) Sufficiency (existence). Suppose $f \in \mathcal{E}_a$ and $\mathcal{F}_1 f(0) = 0$. We let $g(y) = \mathcal{F}_1 f(y)$. Since $g \in \mathcal{E}_a$, $g(y) = \sum_{m=1}^{\infty} (1/m!) D^m g(0) y^m$ (since $g(0) = 0$). It is easy to see that

$$v(y) = \mathcal{F}_1 u(y) = v(0) - \sum_{m=1}^{\infty} (1/m^k m!) D^m g(0) y^m$$

solves equation (1.3). Consequently,

$$u(x) = \mathcal{F}_1^{-1} v(x) = v(0) - \mathcal{F}_1^{-1} \left(\sum_{m=1}^{\infty} (1/m^k m!) D^m g(0) y^m \right) (x) \quad (2.3)$$

solves $-R^k u(x) = f(x)$.

- 3) Uniqueness. Observe that if $h \in \mathcal{E}_a$, then $\tilde{R}h = 0$ iff $h \equiv a$ const. It follows that if W_1, W_2 are two solutions for $-R^k u(x) = f(x)$ such that $W_1, W_2 \in \mathcal{E}_a$ and we let $H = W_1 - W_2$, then we must have $\tilde{R}H = 0$; hence $\tilde{R}(\mathcal{F}_1 H) \equiv 0$.

According to our observation $\mathcal{F}_1 H \equiv C_1$, a constant. Therefore $H \equiv \mathcal{F}_1^{-1}(C_1) \equiv C_1$.

Theorem (2.3): Let $\alpha, \beta, \alpha', \beta'$ be nonzero complex numbers. Then in order that $\mathcal{F}_{\alpha', \beta'}(\mathcal{F}_{\alpha, \beta} f)(z) = f(z)$

for all f in \mathcal{E}_a it is necessary and sufficient that

$$\beta \beta' = 1 \quad \text{and} \quad (\beta \alpha')^2 + \alpha^2 = 0. \quad (3.3)$$

Proof: By Corollary (4.2.6) [114], we see that, for any $f \in \mathcal{E}_a$,

$$\begin{aligned} \mathcal{F}_{\alpha', \beta'}(\mathcal{F}_{\alpha, \beta} f)(z) &= \int_B \int_B f(\alpha x + \beta(\alpha' y + \beta' z)) p_1(dx) p_1(dy) \\ &= \int_B f(\beta \beta' z + \sqrt{\alpha^2 + (\beta \alpha')^2 y}) p_1(dy). \quad (4.3) \\ &= \int_B f(z) p_1(dy). \end{aligned}$$

Now, it becomes obvious that the theorem follows immediately from (4.3).

IV. CONCLUSION

There the Fourier-Wiener transform was defined on a class of mean exponential type analytic functional on the classical Wiener space \mathcal{C} .

As for as the applications of the Fourier-Wiener transform are concerned we find that it is desirable to enlarge the domain of definition of the Fourier-Wiener transform as much as possible (so that it at least also contains a large class of functions other than cylinder functions).

It was shown that the Fourier-Wiener transform is a useful tool for solving differential equations on infinite dimensional spaces.

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