A Theoretical Approach on Generating Functions for Some Polynomials and Special Functions by Using Lie Laplace Transformation

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Abstract— The process of obtaining generating functions takes place by many methods. Lie group and integral transform are considered of the recent methods for obtaining such kind of generating relations. The aim of this study is to obtain generating functions for some polynomials and special functions by using Lie group theoretic approach and Laplace transform. Most of these generating relations are considered as generalizations of previous results that have been obtained before. This study comprises some new generating functions for some polynomials such as Laguerre, Legendre, Hermite, Jacobi, Gegenbauer and Bessel polynomials. Also generating functions for some special functions of three variables such as Exton's functions have been established. The main methods used in our work for obtaining the generating relations are Lie group theoretic method and Laplace transform.

Keywords: Lie Laplace Transformation, Polynomials and Special Functions

I. Introduction

In 1953 Courant and Hilbert [11] were looking into the applications of ordinary differential equations to the field of physics and they came to across the special functions. Also at the same time Morse and Feshbach were studying the applications of Special functions in the field of physical sciences. Hence the development into field of Special functions took place. It is commonly accepted that special functions are those functions that are useful in describing and solving problems.Paul Turan has remarked that special functions go back a very long way in history. Most of the mathematicians of the 18th and 19th century such as Euler, Legendre, Laplace, Gauss, Kummer, Riemann and Ramanujan have contributed to the theory of special functions. The special functions have been and still are studied because of their usefulness in, and interaction with other branches of physics and mathematics, such as number theory, combinatorics and computer algebra and representation theory. The interested reader is referred to the excellent books by Andrews, Rao, Rose and Rainville [2]. The Lie theoretic method is considered one of the recent methods for obtaining generating functions. It was Weisner who first exhibited the group theoretic significance of generating functions for hypergeometric, Hermite and Bessel functions. After that, Miller and McBride present Weisner's method in a systematic manner and thereby lay its firm foundations. Miller also extended Weisner's theory by relating it and connecting to the factorization method, originated by Schr¨odinger and also expanded it by relating to Infield and Hull [12]. Lie algebraic characterizations of two-variables Horn functions have been studied by Kalnins, Onacha and Miller. In the process they evolve a method for obtaining generating functions by expanding a two-variable Horn function in terms of one-variable hypergeometric functions.

II. Definitions and Notations

The definitions and important properties of such elementary functions as Gamma and Beta functions and then precede to the hypergeometric functions in one, two and more variables, which pervade the bulk of the thesis.

III. Laplace Transformation

The Laplace transform is a widely used integral transform with many applications in physics and engineering. It is denoted by L {f(t)}. If f(t) is real valued function defined for $t \geq 0$ and

$$\int_0^{\infty} e^{-ct} |f(t)| dt < \infty$$

where c is positive real number, then we can find the Laplace transform of f(t). It is very clear that this transformation is linear. Outcome of this transformation is generally denoted by F(s) with a complex argument and respective pairs of f(t) and F(s) are matched in tables. The Laplace transform has the useful property that many relationships and operations over the originals f(t)

$$L\{f(t)\} = \int_0^{\infty} \exp(-st) f(t) dt$$
$$= F(s), \quad Re(s) > 0,$$

correspond to simpler relationships and operations over the images F(s). It is named after Pierre-Simon Laplace, who introduced the transform in his work on probability theory. Laplace transform which is given by and inverse

$$L^{-1}\{f(s)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp(st) F(s) ds$$
$$= F(t).$$

and inverse Laplace transform given by

$$L\{t^{\lambda-1}{}_pF_q[\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; zt]\}$$
$$= \frac{\Gamma(\lambda)}{s^{\lambda}}{}_{p+1}F_q\left[\lambda, \alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; \frac{z}{s}\right]$$
$$Re(\lambda) > 0, \; p \leq q, \quad Re(s) > 0 \quad if\ p < q;$$
$$Re(s) > Re(z) \quad if\ p = q.$$

Erdlyi in [9, 10] gave some operational relations on the generalized hyper geometric function. These operational relations are

$$L^{-1}\left\{s^{-\lambda}{}_pF_q\left[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;\frac{z}{s}\right]\right\} = \frac{t^{\lambda-1}}{\Gamma(\lambda)}{}_pF_{q+1}\left[\alpha_1,\ldots,\alpha_p;\lambda,\beta_1,\ldots,\beta_q;zt\right], \quad Re(\lambda)>0,\ p\leq q+1. \tag{i}$$

$$L^{-1}\left\{s^{-\lambda}{}_pF_q\left[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;\frac{z}{s}\right]\right\} = \frac{t^{\lambda-1}}{\Gamma(\lambda)}{}_pF_{q+1}\left[\alpha_1,\ldots,\alpha_p;\lambda,\beta_1,\ldots,\beta_q;zt\right], \quad Re(\lambda)>0,\ p\leq q+1. \tag{ii}$$

Using the preceding relations, Srivastava [5], established some interesting bilinear generating functions for 2F1 and r+1Fs.

IV. GENERATING FUNCTION

A generating function plays an important role in the investigations of various properties of the sequences and polynomials. It may be used to define a set of functions, to determine a differential recurrence relation or a pure recurrence relation, to evaluate certain integrals, etc. More detail about the generating functions may be referred from Srivastava [5] or McBride [6]. We define a generating function for a set of functions as follows

A. *Definition 1.1*

Let F(x, t) be a function that can be expanded in power of t such that

$$F(x,t)=\sum_{n=0}^{\infty}C_n f_n(x)t^n \ ,$$

where Cn is a function of n that may contains the parameters of the set {fn(x)}, but is independent of x and t. Then F(x, t) is called a generating function or linear generating function of the set {fn(x)}.

B. *Definition 1.2*

Multi variables generating function is defined as

$$F(x_1,\ldots,x_r,t)=\sum_{n=0}^{\infty}C_n f_n(x_1,\ldots,x_r)t^n$$

The Legendre Polynomials {Pn(x)} were introduced by Legendre in (1785). He defined them by means of the generating relation:

$$(1-2xt+t^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}P_n(x)t^n.$$

When α= β= 0, the polynomial in (1.4.4) becomes the Legendre polynomial.

The special case of Jacobi polynomial when β= αis called an Ultra-spherical Polynomial and is denoted P(α,α) n (x). Further, in development of orthogonal polynomials, in (1874),

Gegenbauer generalized the Legendre polynomials and used the notation {Cvn (x)} for the set which satisfies the generating relation

$$\exp(2xt-t^2)=\sum_{n=0}^{\infty}\frac{1}{n!}H_n(x)t^n$$

The Hermite polynomials were defined by McBride as well as by Rainville by means of generating relations.

C. *Theorem 1.1*

If there exists a generating function of the form

$$G(x,u,w)=\sum_{n=0}^{\infty}a_n w^n L_n^{(\alpha)}(x)P_m^{(n,\beta)}(u),$$

Then

$$\exp(-wx)(1-wt)^{-(1+\beta+m)}(1+w)^{\alpha}G\left(x(1+w),\tfrac{u+wt}{1-wt},\tfrac{w}{1-wt}\right) = \sum_{n,p,q=0}^{\infty}a_n w^{n+p+q}\frac{(1+n)_p(1+n+\alpha+m)_q}{p!q!}L_{(n+p)}^{(\alpha-p)}(x)P_m^{(n+q,\beta)}(u)t^q.$$

(2

Let us carry forward with the following linear partial differential operators, which has beenreferred from

$$R_1=xy^{-1}z\frac{\partial}{\partial x}+z\frac{\partial}{\partial y}-xy^{-1}z,$$

and

$$R_2=(1+u)t\frac{\partial}{\partial u}+t^2\frac{\partial}{\partial t}+(1+\beta+m)t.$$

So that

$$R_1[y^{\alpha}z^nL_n^{(\alpha)}(x)]=(1+n)L_{(n+1)}^{(\alpha-1)}(x)y^{(\alpha-1)}z^{(n+1)},$$

and

$$R_2[t^nP_m^{(n,\beta)}(u)]=(1+n+\beta+m)P_m^{(n+1,\beta)}(u)t^{(n+1)}.$$

Now, we consider generating function and replacing there w by wtz; and then multiplying both sides by yα, we get,

$$\exp(wR_1)f(x,y,z)=\exp\left(\frac{-wxz}{y}\right)f(x+wxy^{-1}z,y+wz,z),$$

and

$$\exp(wR_2)f(u,t)=(1-wt)^{-(1+\beta+m)}f\left(\frac{u+wt}{1-wt},\frac{t}{1-wt}\right).$$

Operating exp(wR1), exp (wR2) on both sides of , we have

$$y^{\alpha}G(x,u,wtz)=y^{\alpha}\sum_{n=0}^{\infty}a_n(wtz)^nL_n^{(\alpha)}(x)P_m^{(n,\beta)}(u).$$

the left hand side of can be simplified as

$$\exp(wR_1)\exp(wR_2)\left[y^{\alpha}G(x,u,wtz)\right] = \exp(wR_1)\exp(wR_2)\sum_{n=0}^{\infty}a_nL_n^{(\alpha)}(x)y^{\alpha}P_m^{(n,\beta)}(u)(wtz)^n.$$

Finally substituting z/y = 1, we obtain bilateral generating function for generalized modified Laguerre and Jacobi polynomials.

This completes

$$\exp(-wx)(1-wt)^{-(1+\beta+m)}(1+w)^{\alpha}G\left(x+wx, \tfrac{u+wt}{1-wt}, \tfrac{w}{1-wt}\right) =$$

$$\sum_{n,p,q=0}^{\infty} a_n w^{n+p+q} \frac{(1+n)_p (1+n+\beta+m)_q}{p!q!} L_{n+p}^{(\alpha-p)}(x) P_m^{(n+q,\beta)}(u) t^q.$$

This completes the proof of the theorem

V. CONCLUSION

We use the Laplace integral representations of Exton's functions given to determine new generating relations between Lagurree and Jacobi Polynomials.

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