Non-Linear Numerical Schemes for Exact Solutions of Initial Value Problems

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Abstract— In this paper, two first order convergent numerical schemes having nonlinearity in their nature have been extensively explored for their possible exact application upon initial value problems in ordinary differential equations. These two schemes have been Taylor expanded for the purpose of getting their first principal term of the local truncation error which has later been equated to zero in order to get the general form of the underlying ordinary differential equation. In this way, it has been shown that some forms of initial value problems can be obtained which produce exact solutions when these nonlinear schemes under consideration are employed on them. The obtained tabular results in the section of numerical experiments represent the exact solution obtained through the schemes. The technical computing software MAPLE 2016 running under 64-bit operating system with Windows 7 has been used to computations outputs.

Key words: Initial Value Problems; Local Truncation Error; Principal Term; Non-Linear Schemes; Taylor Series

I. INTRODUCTION

The study of ordinary differential equations is a vast mathematical domain which is nearly related to both pure mathematical research and real engineering world applications. Many mathematical expressions and identities of physical laws are represented in terms of ordinary differential equations. Greatly rapid development in the period of 20th century in the fields of physical and natural sciences led to applications of ordinary differential equations in fields like biology, chemistry, medicine, population dynamics, genetic engineering, economy, social sciences and others as well [1-4]. At such advanced level; mathematical models based upon ordinary differential equations require sophisticated schemes to be solved efficiently. At the same time, real world problems have been explored and they have continued to be a great challenge for applied mathematicians and engineers to look for new schemes for the acceptable solutions to such models. In the field of numerical analysis of ordinary differential equations; various schemes of different nature exist. Some are linear whereas others are nonlinear; some are explicit whereas others are implicit; some are single-step whereas others are multi-step; some have smaller stability regions whereas others have larger ones; some are faster in convergence whereas others are slow. It shows that one can have many options to choose a suitable a numerical scheme to serve the purpose. However, researchers have to take care of the errors (local and global) associated with such schemes. Explicit linear numerical schemes while admitting smaller step size contribute into the accumulation of round off errors and thus may pollute our approximate solution for longer integration intervals. In addition to this, they are always conditionally stable making them a little less popular in comparison to other robust schemes having larger stability regions and non-conditionally stable characteristics. While taking large step sizes in linear explicit numerical schemes, one loses some rapid varying properties of the system and also stability of the scheme is highly affected [5-8].

Taylor series expansion has been widely a well-known numerical approach to deal with initial value problems for seeking their approximate solutions but having disadvantage of computing higher successive derivatives of given slope function make it less popular among the community of researchers nowadays. Another reason is that number of numerical schemes have been developed (many based upon the Taylor’s expansion itself) to find the approximate solution of initial value problems. Generally the efficiency of any numerical scheme depends on properties of convergence of the scheme and at least its stability properties [9-12].

Author in [13] proposed a scheme of numerical integration which is suitable for solving initial value problems having oscillatory or exponential solutions. In [14-17]; nonlinear explicit one-step integration scheme for singular autonomous and non-autonomous initial value problems have been developed. In [18], the author has determined general form of ordinary differential equations for which standard numerical schemes including explicit Euler, implicit Euler, trapezoidal rule, second and third-order Taylor, van Niekerk’s second and third order rational schemes have produced exact solution to the obtained initial value problems. The current research paper is a motivation sought from [18] in which the authors in the present study have tested mostly nonlinear type of numerical schemes to yield exact (analytical) solutions of some general forms of initial value problems.

II. PROBLEM DESCRIPTION & METHODOLOGY

Suppose a well posed initial value problem of the form is given as

$$\frac{dy(x)}{dx} = y'(x) = f\left(x, y(x)\right);\ y(x_0) = y_0 \tag{2.1}$$

Nowadays; non-linear scheme are being developed rapidly to solve the problem of type (2.1) and it is very interesting to develop some general form of ordinary differential equations to achieve exact solutions for the underlying numerical schemes. In order to achieve our goal; two nonlinear schemes with first order convergence have been chosen to be employed on some standard form of initial value problems for their possible exact solution. These chosen nonlinear numerical schemes have then be Taylor expanded to get the principal term of the local truncation error. We would then equate the principal term of the local truncation error of the schemes to zero. The non-linear scheme chosen to serve this purpose are shown by (2.3) and (2.4). Furthermore, to obtain the required results we would use the convergent Taylor's series of a smooth function $y = f(x)$ about $x = 0$ as shown by (2.2) below.

$$y(x_n + h) = y(x_n) + hy'(x_n) + \frac{h^2}{2!}y''(x_n) + \frac{h^3}{3!}y'''(x_n) + O(h^4) \quad (2.2)$$

A. First Order Nonlinear Scheme

$$y_{n+1} = \frac{y_n^2}{y_n - hf(x_n, y_n)} \quad (2.3)$$

This scheme is nonlinear first order convergent as proposed in [13].

B. First Order Nonlinear Autonomous Scheme

$$y_{n+1} = \frac{y_n^2}{y_n - hf(y_n)} \text{ where } f(y_n) = y_n' \quad (2.4)$$

This scheme is also nonlinear first order convergent as proposed in [15].

The necessary condition for solving ordinary differential equations exactly is that its principal term of the local truncation error must be equal to zero and this term may involve higher derivatives of first and second order. Further, it may be a linear or nonlinear differential equation which is then solved to find the general solution using Maple 2016 software which is also used to perform other numerical simplifications in the present research work. The principal term of the local truncation error for the schemes listed below have been derived and then equated to zero for further study.

The Local Truncation Error (LTE) of (2.3) has been derived using Taylor series expansion as discussed below:

$$\text{LTE} = y(x_n + h) - y_{n+1}$$

$$\text{LTE} = y(x_n) + hy'(x_n) + \frac{h^2}{2!}y''(x_n) - \left\{\frac{y_n^2}{y_n - hy_n'}\right\} + O(h^3)$$

$$\text{LTE} = y(x_n) + hy'(x_n) + \frac{h^2}{2!}y''(x_n) - y_n\left\{\left(1 - \frac{hy_n'}{y_n}\right)^{-1}\right\} + O(h^3)$$

Using binomial theorem, one obtains

$$\text{LTE} = y(x_n) + hy'(x_n) + \frac{h^2}{2!}y''(x_n) - y_n\left\{1 + \left(\frac{hy_n'}{y_n}\right) + \left(\frac{hy_n'}{y_n}\right)^2\right\} + O(h^3)$$

$$\text{LTE} = h^2\left(\frac{y''(x_n)}{2!} - \frac{y'^2(x_n)}{y(x_n)}\right) + O(h^3)$$

The principal term of the local truncation error for the scheme (2.3) is finally expressed as:

$$\text{LTE} = h^2\left(\frac{y''(x_n)}{2!} - \frac{y'^2(x_n)}{y(x_n)}\right) + O(h^3) \quad (2.5)$$

Here one obtains the second order nonlinear ordinary differential equation as given below after equating principal term of this local truncation error to zero.

$$y(x_n)y''(x_n) = 2y'^2(x_n) \quad (2.6)$$

We have determined the general exact solution of the second differential equation (2.6) by Maple 2016 software as given below:

$$y(x) = \frac{A}{x + B} \quad (2.7)$$

The associated first order ordinary differential equation of (2.7) is obtained as

$$f(x, y(x)) = \frac{-A}{(x + B)^2}$$

where A and B are arbitrary integration constants. Solving (2.7) for A, B and x respectively; the following three ordinary differential equations have been obtained:

$$f_1\left(x, \frac{(x+B)y(x)}{(x+B)}\right) = \frac{-(x+B)y(x)}{(x+B)^2}$$

$$f_1(x, y(x)) = \frac{-y(x)}{(x+B)}$$

$$f_2\left(x, \frac{A}{\left(\frac{A}{y(x)} - B\right) + B}\right) = \frac{-A}{\left(\frac{A}{y(x)} - B + B\right)^2}$$

$$f_2(x, y(x)) = \frac{-y^2(x)}{A}$$

$$f_3\left(x, \frac{A}{\left(x + \frac{A}{y(x)}\right) - x}\right) = \frac{-A}{\left(x + \frac{A}{y(x)} - x\right)^2}$$

$$f_3(x, y(x)) = \frac{-(y(x))^2}{A}$$

Two general forms (linear and nonlinear) have been produced from the above obtained ordinary differential equations and these general forms are given below:

$$(x + B)y'(x) + y(x) = 0 \quad (2.8)$$

$$Ay'(x) + y^2(x) = 0 \quad (2.9)$$

Applying the scheme (2.3) on equation (2.8), one gets

$$y_{n+1} = y_n^2\left[y_n - h\left(\frac{-y_n}{x_n + B}\right)\right]^{-1}$$

$$y_{n+1} = \frac{y_n(x_n + B)}{(x_n + B + h)} \quad (2.10)$$

And applying the scheme (2.3) on equation (2.9), one gets

$$y_{n+1} = y_n^2\left[y_n - h\left(\frac{-y_n^2}{A}\right)\right]^{-1}$$

$$y_{n+1} = \frac{y_n A}{A + hy_n} \quad (2.11)$$

These two numerical schemes obtained in (2.10) and (2.11) exactly coincide with the exact solution of (2.8) and (2.9) respectively.

Now, we carry out the calculation for the nonlinear schemes given in (2.4). The Local Truncation Error (LTE) of (2.4) has been derived using Taylor series expansion as discussed below:

$$\text{LTE} = y(x_n + h) - y_{n+1}$$

$$\text{LTE} = y(x_n) + hy'(x_n) + \frac{h^2}{2!}y''(x_n) - \left\{\frac{y_n^2}{y_n - hy'_n}\right\} + O(h^3)$$

$$\text{LTE} = y(x_n) + hy'(x_n) + \frac{h^2}{2!}y''(x_n) - y_n\left\{\left(1 - \frac{hy'_n}{y_n}\right)^{-1}\right\} + O(h^3)$$

Using binomial theorem, one gets

$$\text{LTE} = y(x_n) + hy'(x_n) + \frac{h^2}{2!}y''(x_n) - y_n\left\{1 + \left(\frac{hy'_n}{y_n}\right) + \left(\frac{hy'_n}{y_n}\right)^2\right\} + O(h^3)$$

$$\text{LTE} = h^2\left(\frac{y''(x_n)}{2!} - \frac{y'^2(x_n)}{y(x_n)}\right) + O(h^3)$$

$$\therefore\ f(y_n) = y'_n = f \text{ and } f_x + ff_y = y''_n$$

where $f_x = 0$ because given scheme is autonomous.

The principal term of the local truncation error for the scheme (2.4) is finally expressed as:

$$\text{LTE} = h^2\left(\frac{1}{2}ff_y - \frac{f^2}{y_n}\right) + O(h^3)$$

Here one obtains the second order nonlinear ordinary differential equation as given below after equating principal term of this local truncation error to zero.

$$\frac{1}{2}ff_y - \frac{f^2}{y_n} = 0 \qquad (2.12)$$

Once again, the general solution of the above differential equation by Maple is as given below:

$$y(x) = \frac{A}{x+B} \qquad (2.13)$$

The associated first order ordinary differential equation of (2.13) is obtained as

$$f(x, y(x)) = \frac{-A}{(x+B)^2}$$

Where A and B are arbitrary integration constants. Solving (2.13) for A, B and x respectively; the three ordinary differential equations obtained are found exactly the way as above. Two general forms (linear and nonlinear) have been produced from the above obtained ordinary differential equations as given below:

$$(x+B)y'(x) + y(x) = 0$$

and

$$Ay'(x) + y^2(x) = 0$$

Since these are the same equations as obtained in earlier case therefore one obtains

$$y_{n+1} = \frac{y_n(x_n+B)}{(x_n+B+h)} \quad \text{and} \quad y_{n+1} = \frac{y_n A}{A + hy_n}$$

They exactly coincide with the exact solution of (2.8) and (2.9) respectively.

III. NUMERICAL EXPERIMENTS

Looking at the above developed strategy, both the nonlinear schemes under consideration; that is., the schemes (2.3) and (2.4) have yielded the schemes of the form (2.10) and (2.11) which will ultimately produce exact solution for the general form of ordinary differential equations as given by (2.13) and (2.14). In order to test the obtained schemes, a few initial value problems of varying nature are discussed below.

A. *Problem 01*

$$(x+1)y'(x) + y = 0$$

$$y(0) = 1,\ h = 0.5,\ x \in [0,5]$$

Exact solution: $y = \dfrac{1}{(x+1)}$

x	Exact	Scheme (2.10)
1	5.0000e-01	5.0000e-01
2	3.3333e-01	3.3333e-01
3	2.5000e-01	2.5000e-01
4	2.0000e-01	2.0000e-01
5	1.6667e-01	1.6667e-01

Table 1: Scheme (2.10) for Problem 1

B. *Problem 02*

$$y'(x) + y^2(x) = 0$$

$$y(0) = 1,\ h = 0.5,\ x \in [0,5]$$

Exact solution: $y = \dfrac{1}{(x+1)}$

x	Exact	Scheme (2.11)
1	5.0000e-01	5.0000e-01
2	3.3333e-01	3.3333e-01
3	2.5000e-01	2.5000e-01
4	2.0000e-01	2.0000e-01
5	1.6667e-01	1.6667e-01

Table 2: Scheme (2.11) for Problem 2

C. *Problem 03*

$$(x+2)y' + y = 0\ ;\ y(0) = 1$$

$$y(0) = 1,\ h = 0.5,\ x \in [0,5]$$

Exact solution: $y = \dfrac{2}{(x+2)}$

x	Exact	Scheme (2.10)
1	6.6667e-01	6.6667e-01
2	5.0000e-01	5.0000e-01
3	4.0000e-01	4.0000e-01
4	3.3333e-01	3.3333e-01
5	2.8571e-01	2.8571e-01

Table 3: Scheme (2.10) for Problem 3

D. *Problem 04*

$$3y' + y^2 = 0$$

$$y(0) = 3,\ h = 0.5,\ x \in [0,5]$$

Exact solution: $y = \dfrac{3}{x+1}$

x	Exact	Scheme (2.11)
1	1.5000e+00	1.5000e+00
2	1.0000e+00	1.0000e+00
3	7.5000e-01	7.5000e-01
4	6.0000e-01	6.0000e-01
5	5.0000e-01	5.0000e-01

Table 4: Scheme (2.11) for Problem 4

IV. CONCLUSION

The main focus of this research work is to compute general form of initial value problems which can exactly be solved by explicit one-step non-linear schemes under consideration. In the above given examples; it is clear that the ordinary differential equations are easy to solve by developed schemes compared with the exact solution. The tables above show that the answers obtained by the schemes exactly match with those of the analytical solutions. Main contribution of the present research work is demonstrated by the obtained schemes shown in (2.10) and (2.11) which sought exact results upon their applications on certain forms of the initial value problems. Some higher order nonlinear schemes will be explored in future to be used on initial value problems to get the exact solutions.

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