Dependency between Exchange Rate & Gold Price via Copula-DCC-GARCH Approach

Emre Yildirim¹, Mehmet Ali Cengiz²
¹,²Department of Statistics
¹,²University of Ondokuz Mayis, Samsun, Turkey

Abstract— The dependency structure among financial assets is the main source of uncertainties in the economy. In this study, the dependency structure between dollar and gold returns is modelled via the Copula-DCC-GARCH approach. This method provides rather effective results in modelling the characteristics of financial assets such as asymmetry and heavy tail. It also allows the determination of many dependence structures such as tail dependency and non-linear dependencies. In the first stage of the study, standardized residues are obtained by applying the DCC-GARCH method to returns of dollar and gold. Then, the dependency structure between gold and dollar returns is modelled by means of dynamic copulas. As a result of the study, it was found that Student’s t Copula models the best dependency between dollar and gold returns, and the dependency is not constant but change over time as well as in the process of the extreme events in the financial markets, the dollar and gold prices tend to co-movement.

Key words: Dependence Structure, Dynamic Copula, Exchange Rate, Gold

I. INTRODUCTION

There exists a large number of relationship between variables. Modelling accurately these relations is commonly studied in many discipline such as statistics and econometrics. Features of variables related have crucial role in determining dependency between variables. Financial variables are extensively investigated by economists and researchers and investors. Modelling correctly the dependency provides investors to be led to profitable investments as it helps investors take measures on investment plan by allowing investors for estimating possible losses. In literature there exists some methods used to model dependency between financial variables. It is necessary for satisfying some assumption such as normality for these method. However, it is determined that financial variables are skewed and leptokurtic in consequence of empirical studies. In this case, modelling the dependency via classical techniques such as MGARCH may give misleading results. Copulas which do not need the assumption such as normality in modelling the dependency between variables and allow for modelling various dependence structure such as tail dependency have become a popular tool in econometrics and finance. Copulas which construct a link between a multivariate distribution function and it’s univariate marginals are function defined in [0,1] interval. Since copulas make possible to model the dependency related variables irrespective of marginal distributions, they are extensively used in many disciplines. While for detailed information on copulas, Nelsen (2007), Joe (2014), Durante et al. (2010) can be viewed, for copula information on finance, Cherubini et al. (2011) can be studied. Wu et al. (2012) investigate the dependency between oil price and exchange rate by means of dynamic and static copula-GARCH model. As a result of the study, it is found that dynamic Student’s t copula is better model than static copula and other method in terms of out of sample performance. Chollete et al. (2009) construct copula-regime switching model in determining asymmetric dependency between returns of international financial markets via Gaussian and vine copulas and it is determined that vine copulas give better results than Gaussian copula. Wang et al. (2011) view dependence structure between China markets and international major markets and increasing effect of China on global economy using copula-GARCH. They found that China markets are highly correlated with Japan and Pacific markets in consequence of the study. Wen et al. (2012) investigate whether there is contagion effect between energy and stock markets during recent financial crisis using time varying copula. They found that there is an increasing dependency between crude oil price and stock market and contagion effect in USA markets are higher than China markets. Wu and Lin (2014) search dependence structure between stock and bond returns via copula-based GARCH model. They determine that GJR-GARCH model with Student’s t copula outperform other models. In this paper, dependence structure between dollar and gold price is investigated by means of copula-DCC-GARCH model. This method used models conditional correlation between variables via DCC-GARCH and conditional dependency via time varying copulas. Thus, it allows for comprehensive dependence modelling for non-normal multivariate distributions. Dependence modelling is performed by taking into account characteristics such as skew and kurtosis of financial variables by means of the proposed model. Autocorrelation and heteroscedasticity issues typically encountered are handled with the help of DCC-GARCH approach. After standardized residuals obtained from this approach are transform into uniform distribution via empirical cumulative distribution function which is semi parametric method, these uniform variables are used as input variables for copula modelling. Selection of the best fitted time varying copula model is determined based on AIC and BIC information criteria. Finally, dependence structure between dollar and gold is modelled by means of the best fitted copula selected. In this paper, it is found that dynamic models are better than static one in modelling dependency between variables. Dependence structure between dollar and gold prices changes considerably over time. In addition, there is a symmetric dependence as well as tail dependency between dollar and gold price. It means that dollar and gold prices tend to co-movement during extreme market events.

The remainder of this paper is organized as follows. In section 2, the theoretical background of copula-DCC-GARCH method is offered. In section 3, datasets employed are described and empirical findings are discussed while in section 4 results inferred from this paper is presented.
II. METHODOLOGY

In this section theory of copula and dynamic copula models in modelling dependency are introduced. Then, dynamic conditional heteroscedasticity (DCC) and generalized autoregressive conditional heteroscedasticity (GARCH) models are presented.

A. Theory of Copula

Copula is a distribution function with following properties:

1) For all u,v
\[
C(u,0) = 0 = C(0,v)
\]
\[
C(u,1) = u \text{ and } C(1,v) = v \tag{1}
\]

2) For all \(u_1, u_2, v_1, v_2\) satisfying \(u_1 \leq u_2\) \& \(v_1 \leq v_2\)
\[
C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0 \tag{2}
\]

Sklar’s theorem indicates the role of copulas in relationship between multivariate distribution functions and their univariate marginals. Let \(H\) be joint distribution function with \(F\) and \(G\) marginals. Then, all \((x,y)\) in \(R\), there is a \(C\) copula and is defined as follows:
\[
H(x,y) = (F(x), G(y)) \tag{4}
\]

C Copula is a unique if \(F\) and \(G\) are continuous. On the other hand, \(H\) is a joint distribution function with \(F\) and \(G\) marginals if \(F\) and \(G\) are distribution functions. Probability density function of any multivariate probability distribution is represented via Sklar’s theorem as follows:
\[
f(x_1, \ldots, x_n) = \frac{\partial C(u_1, \ldots, u_n)}{\partial u_1 \ldots \partial u_n} = \frac{\partial C(u_1, \ldots, u_n)}{\partial x_1} \prod_{i=1}^{n} f_i(x_i) \tag{5}
\]

Where \(C\) and \(f_i\) represent density function of the copula and marginal density function, respectively. This property provides significant flexibility in constructing multivariate distributions since it is not necessary that marginals and copula come from similar distribution.

B. Dynamic Copula Models

Dynamic (conditional) copula models introduced by Patton (2006) is known as models considering the changes over time in dependence structure. Proposed models assume that copula parameters which drives dependency vary in time. Dynamic copula models widely used is presented as follows:

1) Time Varying Gaussian Copula

Gaussian copula which is powerful tool in modelling symmetric dependence has \(\rho_t\) parameter defined as follows:
\[
\rho_t = \Lambda \left( \omega_0 + \beta (\tau_{t-1}) + \alpha \sum_{i=1}^{q} \Phi^{-1}(u_{t-j}) \Phi^{-1}(v_{t-j}) \right) \tag{6}
\]

Where \(\Phi^{-1}\) is the inverse of standard normal cumulative distribution function? In addition, \(\Lambda(x) = (1 - e^{-x})(1 + e^{-x})^{-1}\) is modified logistic transformation and it guarantees the value of \(\rho_t\) parameter estimation to keep (-1,1) interval.

2) Time Varying Student’s t Copula

Student’s t copula can model asymmetric dependency as well as symmetry. It has \(\rho_t\) parameter described as follows:
\[
\rho_t = \Lambda \left( \omega_0 + \beta (\tau_{t-1}) + \alpha \sum_{i=1}^{q} t^{-1}_v(u_{t-j}) t^{-1}_v(v_{t-j}) \right) \tag{7}
\]

Where \(t^{-1}_v\) is the inverse of Student’s t cumulative distribution function? Moreover, \(\Lambda(x) = (1 - e^{-x})(1 + e^{-x})^{-1}\) is modified logistic transformation and it assures \(\rho_t\) parameter estimation to keep (-1,1) interval.

3) Time Varying Rotated Gumbel Copula

Time varying rotated Gumbel copula uses \(\tau_t = 1 - \theta_t\) transformation and it is characterized as follows:
\[
\tau_t = \Lambda \left( \omega_0 + \beta \tau_{t-1} + \alpha \sum_{i=1}^{q} u_{t-j} - v_{t-j} \right) \tag{8}
\]

Where, \(\Lambda(x) = (1 + e^{-x})^{-1}\) is a logistic transformation and it ensures the value of \(\tau_t\) parameter estimation to keep (-1,1) interval.

4) Time Varying Symmetrized Joe Clayton Copula

Time varying symmetrized Joe Clayton copula introduced by Patton (2006) can model lower and upper tail dependencies defined as follows:
\[
\lambda^U_t = \Lambda \left( \omega_0 + \beta \lambda^U_{t-1} + \alpha \sum_{i=1}^{q} u_{t-j} - v_{t-j} \right) \tag{9}
\]
\[
\lambda^L_t = \Lambda \left( \omega_0 + \beta \lambda^L_{t-1} + \alpha \sum_{i=1}^{q} u_{t-j} - v_{t-j} \right) \tag{10}
\]

Here, \(\Lambda(x) = (1 + e^{-x})^{-1}\) that guarantees \(\lambda^U_t\) and \(\lambda^L_t\) parameters to keep \((0,1)\) interval is a logistic transformation. Copula-DCC-GARCH approach comprises of two steps. First, DCC-GARCH model is implemented to model conditional correlation and conditional volatility and standardized residuals are acquired from this model. Second, standardized residuals are transformed into \([0,1]\) interval by empirical cumulative distribution function. Thus, uniform inputs which is necessary for copula models are obtained and the best fitted time varying copula model is estimated. Finally, dependence structure between dollar and gold returns is modelled via estimated copula.

C. Copula Estimation Methods

Copula estimation methods are divided into three parts as parametric, semi-parametric and non-parametric. In this study, we will describe semi-parametric estimation methods since it provides some advantages in estimating parameters.

1) Semiparametric Estimation Method

Semiparametric method of copula estimation is called as Canonical Maximum Likelihood (CML). In this method, assuming parametric distributions for marginals is not necessary and empirical cumulative distribution functions are used for marginals. \(F_X(x)\) Empirical distribution function tends to approximate \(F_X(x)\) unknown parametric marginals and it is defined as follows:
\[
F_X(x) = \frac{1}{T+1} \sum_{t=1}^{T+1} 1[x_{s_t} \leq x], i = 1, \ldots n \tag{11}
\]

Where, \(1[x_{s_t} \leq x]\) is called as indicator function. CML method is applied in two steps:
1) Step: Data sets $X = (X_1, ..., X_n)$ is transformed into uniform variables using empirical distribution function.

$$\hat{U} = (\hat{u}_1, ..., \hat{u}_n) \in \left[ \hat{F}_1(X_1), ..., \hat{F}_n(X_n) \right]$$

Step: copula parameter vector is estimated via empirical distributions obtained from step 1.

$$\hat{\alpha}_{\text{CML}} = \arg \max_a \{ \ln(\hat{u}_1, ..., \hat{u}_n; \alpha) \}$$

CML estimator is described as follows:

$$\theta_{\text{CML}} = (\hat{\alpha}_{\text{CML}})$$

D. DCC-GARCH Model

Time varying correlations are frequently estimated via multivariate conditional heteroscedasticity models. DCC-GARCH approach proposed by Engle (2002) can model time varying conditional correlation as well as conditional volatility. Let $r_{1,t}$ and $r_{2,t}$ be returns of two financial assets at t time. DCC-GARCH model is formulated as follows:

$$(r_{1,t}, r_{2,t}) = \left( \frac{c_1}{c_2}, \frac{\varepsilon_{1,t}}{\varepsilon_{2,t}} \right)$$

Where $c_1$ and $c_2$ are expected values of returns of first and second assets, respectively. $\varepsilon_{1,t} \& \varepsilon_{2,t}$ are error term of returns of first and second assets given $\Omega_{t-1}$ information in turn. $H_t$ is conditional variance-covariance matrix given $\Omega_{t-1}$ information and it is decomposed using Cholesky method as follows:

$$H_t = \begin{pmatrix} h_{1,t} & h_{1,2,t} \\ h_{2,1,t} & h_{2,2,t} \end{pmatrix} = \begin{pmatrix} h_{1,t} & 0 \\ 0 & h_{2,t} \end{pmatrix} \begin{pmatrix} 1 & \rho_t \\ \rho_t & 1 \end{pmatrix} \begin{pmatrix} h_{1,t} & 0 \\ 0 & h_{2,t} \end{pmatrix}$$

In addition, Eq.(16) is defined as $H_t = D_t R_t D_t^t$.

Here, $D_t$ is a diagonal matrix composing of conditional standard deviations of returns and $R_t$ is a conditional correlation matrix of returns, respectively.

1) GARCH Model

Generalized autoregressive conditional heteroscedasticity (GARCH) model is proposed by Bollerslev (1986) to model conditional heteroscedasticity. This model is characterized as follows:

$$h_t^2 = \omega + \sum_{i=1}^{k} \alpha_i \varepsilon_{t-i}^2 + \sum_{l=1}^{k} \beta_l h_{t-l}^2$$

Where, $Q_t$ is a symmetric and positive definite matrix and it is given below:

$$Q_t = (1 - \theta_1 - \theta_2) \hat{Q} + \theta_1 \eta_{t-1} \hat{\eta}_{t-1} + \theta_2 Q_{t-1}$$

Here, $\theta_1 \& \theta_2$ that satisfy $\theta_1 + \theta_2 < 1$ are non-negative values. $\hat{Q}$ is unconditional (time-invariant) covariance matrix of standardized residuals defined in Eq. (21).

$$\eta_{t,t} = \frac{\varepsilon_{t,t}}{\sqrt{h_t}}$$

Thus, conditional correlation matrix of returns series is obtained using dynamic conditional correlation model.

III. DATA & EMPIRICAL RESULTS

A. Data Description

Data set used in this study comprises of dollar and gold daily prices between March 2015 and April 2017 and the number of observation is 799. All price series is demonstrated in Fig. 1. Since all price series have trend, stationary condition for the series is not satisfied. Therefore, returns series are used instead of price series in modelling dependency and return series are calculated as noted below:

$$r_{1,t} = \ln \left( \frac{p_{1,t}}{p_{1,t-1}} \right)$$

Here, $p_{1,t}$ is a price of ith asset at t time. Return series of dollar and gold are shown in Fig. 2 and descriptive statistics of the series are presented in Table 1.

<table>
<thead>
<tr>
<th></th>
<th>Dollar</th>
<th>Gold</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.00057</td>
<td>0.00077</td>
</tr>
<tr>
<td>Std deviation</td>
<td>0.00794</td>
<td>0.00995</td>
</tr>
<tr>
<td>Min</td>
<td>-0.03193</td>
<td>-0.03677</td>
</tr>
<tr>
<td>Max</td>
<td>0.04835</td>
<td>0.07554</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>6.03216</td>
<td>7.70696</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.57036</td>
<td>0.73447</td>
</tr>
<tr>
<td>ADF p-value</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>Jarque Bera p-value</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>Ljung-Box(Q) p-value</td>
<td>0.36960</td>
<td>0.57740</td>
</tr>
<tr>
<td>Ljung-Box(Q^2) p-value</td>
<td>0.00030</td>
<td>0.14720</td>
</tr>
<tr>
<td>Arch LM p-value</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
</tbody>
</table>

Table 1: Descriptive Statistics of Return Series

Averages of daily return series are so close to zero and the values of standard deviation are relatively higher than means. This implies that dollar and gold assets have high volatility. It is determined that gold return is higher than dollar return but gold is more volatile investment tool than dollar. It is found that returns of both financial assets are skewed and kurtosis according to Gaussian distribution and this refers that series are not normally distributed.
Fig. 1: Dollar & Gold Prices, Respectively

Jorqe Bera test is used to determine whether series are normally distributed. According to test result, null hypothesis indicating that series are normally distributed is rejected. Stationary condition for all series are controlled via Augmented Dickey Fuller test and it is found that return series of dollar and gold are stationary. Autocorrelation test is performed for both return series and squares of return series by means of Ljung-Box Q. Results refer that there is no serial correlation for return series but square of gold return series has autocorrelation. This points out that there is a possible volatility clustering in gold return series. Finally, it is investigated whether there are ARCH effects in return series via Engle’s ARCH test and null hypothesis specifying that there is no ARCH effect is rejected. It is concluded that using GARCH models is necessary for the series.

Fig. 2: Return Series of Dollar & Gold, Respectively

B. Empirical Results

Results of DCC-GARCH (1,1) model used to determine dynamic volatility and correlation between dollar and gold are presented in Table 2.

Table 2: Parameter estimations of DCC-GARCH (1, 1) model

<table>
<thead>
<tr>
<th></th>
<th>Dollar</th>
<th>Gold</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega )</td>
<td>0.00002</td>
<td>0.00001</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
</tr>
<tr>
<td>( \beta )</td>
<td>0.12013</td>
<td>0.04665</td>
</tr>
<tr>
<td>( \beta_1 )</td>
<td>(0.0179)</td>
<td>(0.0049)</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>0.65353</td>
<td>0.90433</td>
</tr>
<tr>
<td>( \lambda_1 )</td>
<td>(0.0420)</td>
<td>(0.0111)</td>
</tr>
<tr>
<td>( \nu )</td>
<td>1.12762</td>
<td>1.12132</td>
</tr>
<tr>
<td>( \nu_1 )</td>
<td>(0.0595)</td>
<td>(0.0551)</td>
</tr>
<tr>
<td>( \theta_1 )</td>
<td>5.41150</td>
<td>5.29313</td>
</tr>
<tr>
<td>( \theta_2 )</td>
<td>(0.8582)</td>
<td>(0.9582)</td>
</tr>
<tr>
<td>( \nu_j )</td>
<td>1.12759</td>
<td>1.12132</td>
</tr>
</tbody>
</table>

Table 3: Autocorrelation Test Results of Standardized Residuals

<table>
<thead>
<tr>
<th></th>
<th>Dollar</th>
<th>Gold</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ljung-Box(Q) p-value</td>
<td>0.3357</td>
<td>0.6197</td>
</tr>
<tr>
<td>Ljung-Box(Q^2) p-value</td>
<td>0.0751</td>
<td>0.6031</td>
</tr>
</tbody>
</table>

Table 4: Model Selection for the Best Fitted Copula

<table>
<thead>
<tr>
<th></th>
<th>LogL</th>
<th>AIC</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time varying Gaussian</td>
<td>120.62</td>
<td>237.25</td>
<td>228.16</td>
</tr>
<tr>
<td>Time varying Student’s t</td>
<td>125.61</td>
<td>245.22</td>
<td>231.58</td>
</tr>
<tr>
<td>Time varying Gumbel</td>
<td>81.736</td>
<td>157.47</td>
<td>143.82</td>
</tr>
<tr>
<td>Time varying symmetrized Joe-Clayton</td>
<td>128.04</td>
<td>244.08</td>
<td>216.79</td>
</tr>
</tbody>
</table>

While \( \beta \) parameter of time varying Student’s t copula measures stability degree in dependency over time, \( \alpha \) parameter of the model quantifies variability degree in financial assets has volatility over time and it indicates that selection of DCC-GARCH (1,1) model is appropriate for modelling conditional volatility and correlation. It is determined that coefficients of conditional variance are statistically significant and positive. It is found that coefficients of lagged conditional variance \( \beta_i \) in GARCH model are larger than coefficients \( \alpha_i \) of past return errors. This indicates that there is a fundamental stability effect in volatilities of dollar and gold returns.
dependence structure. As seen in Table 5, stability parameter $\beta$ of Student’s $t$ copula is significant and positive. This refers that stability degree is rather high in dependency between dollar and gold returns over time. Parameter $\alpha$ which is statistically significant indicates that there is a substantial variability in dependence process during time. Positive parameter $\alpha$ refers that the latest information is a meaningful indicator in modelling dynamic dependency between dollar and gold return. It is found that variability effect in dependency over time is dominant than stability effect since Parameter $\alpha$ is larger than parameter $\beta$. Significant parameter estimations of Student’s $t$ copula remark that dependence structure between dollar and gold changes over time and this demonstrates that static copulas may be insufficient to the model dependency between related variables.

<table>
<thead>
<tr>
<th>Estimation of parameters</th>
<th>(S.E.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega$</td>
<td>0.005</td>
</tr>
<tr>
<td></td>
<td>(0.004)</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>7.594</td>
</tr>
<tr>
<td></td>
<td>(3.052)</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.9914</td>
</tr>
<tr>
<td></td>
<td>(0.005)</td>
</tr>
</tbody>
</table>

Table 5: Parameter Estimations of the Best Fitted Copula

Selection of Student’s $t$ indicates that there is a tail dependency as well as symmetric dependency between dollar and gold. This means that dollar and gold markets tend to co-movement when extreme events occur in financial markets. In other words, dollar and gold markets demonstrate similar behaviour in boom and bust. Dependence path between dollar and gold returns is exhibited in Fig. 3. It is determined that dependency between dollar and gold returns evolves dramatically during period selected and this supports the use of dynamic copulas in modelling dependency between related variables. As seen in Fig. 3, dependency ranges from 0.48 to 0.62 and this implies that dynamic correlation between dollar and gold return can give better result than static one. Thus, investors or risk managers can carry out better investment plan with the help of proposed model.

![Fig. 3: Dependence Path between Dollar & Gold Return](image-url)

**IV. CONCLUSION**

In this paper, dynamic dependence structure between returns of dollar and gold is investigated via Copula-DCC-GARCH approach. This method allows for conditional correlation by DCC-GARCH and conditional dependency via a dynamic copula. Thus, time varying dependency between financial markets is extensively analysed. First, conditional correlation between dollar and gold is modelled and then standardized residuals are obtained. Second, the residuals obtained from DCC-GARCH are transformed into uniform distribution and they are used as input variables for copula approach. Dynamic dependency between returns of dollar and gold is modelled via time varying (conditional) copulas. It is found that Time varying Student’s $t$ copula which can model tail dependency as well as symmetric dependency structure is the best fitted model. Moreover, variation effect is higher that stability effect over time in dependency between dollar and gold returns. Dollar and gold returns tend to co-movement when extreme events occur in financial markets.

**REFERENCES**