On Common Fixed-Point for G-Metric space

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Abstract— In this dissertation we discuss about the paper, unique common fixed point theorem for two pair of weakly compatible maps in a complete metric space, which generalizes the result of Brian Fisher by a weaker condition such as weakly compatibility instead of compatibility and contractive modulus instead of continuity of maps.

Key words: Metric Space, Mapping, Continuous, Bounded, Fixed Point

I. Introduction

The objective of this paper is to introduce the notion of semi-compatible maps in D-metric spaces and deduce fixed point theorems through semi compatibility using orbital concept, which improve extend and generalize the results of Ume and Kim, Rhoades and Dhage. All the results of this paper are new.

In this paper we extend and generalize the concept of F-contraction to F- weak contraction and prove a fixed point theorem for F-weak contraction in a complete G-metric space. The article includes a non-trivial example which verifies the effectiveness and applicability of our main result.

II. Main Result

A. *On Common Fixed-Point Theorem for G- Metric Space*

1) *Definition: 2.1*

A **Metric Space** is a set x together with a function d (called a metric or "distance function") which assigns a real number $(x$, y) to every pair x, y X satisfying the properties.

a) (positive) For all , $y \in X$, $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
b) (symmetry) for all $x, y \in X$, $d(x, y) = d(y, x)$.
c) (the triangle inequality) for all $x, y, z \in X$, $d(x, y) \leq d(x, y) + d(y, z)$.

2) *Example 2.2*

Let $X = [0,1]$ and $G(x, y, z) = {}^1 (|x - y| + |y - z| + |x - z|)$ for all $x, y, z \in X$. then (X, G) is a complete G-metric space. Define a mapping $T: X \to X$ by

$$T_x = \begin{Bmatrix} \frac{1}{2}\,; 0 < x < 1 \\ \frac{1}{4}; if\ x = 1 \end{Bmatrix}$$

Proof: Since T is not continuous, therefore it is not an $F-$ contraction for any mapping F a described in Let $F: \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying

(F1) F is strictly increasing. That is, $\alpha < \beta \Longrightarrow (\alpha) < (\beta)$ for all $\alpha, \beta \in \mathbb{R}^+$.

(F2) For every sequence $\{\alpha n\}$ in $\mathbb{R}^+$ we have $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

(F3) There exists a number $\in (0, 1)$ such that $\lim_{n\to\infty} F(\alpha) = 0$

However, for $x, y, z \in [0, 1)$, $z = 1$, we have $(T_x, T_y, T_z) = G\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}\right) = \frac{1}{6} >$ 0 and $\{(x, y, 1), (x, T_x, T_x), (y, T_y, T_y), (1, T_1, T_1)\} >$ $(1, T_1, T_1) = \frac{1}{2}$ Taking $(\alpha) = ln\alpha$, $\alpha \in (0, \infty)$ and $\tau = ln3$, we see that T is an F -weak contraction

3) *Remark 2.3*

Let be an $F-$ weak contraction. Let (X, G) be a $G-$metric space. Then the following are equivalent

a) The sequence $\{x_n\}$ is $G-$ cauchy. For every $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $(xn, xm, xm) < \varepsilon$, for all $n, m \geq k$. we have for all $x, y, z \in X$ with $(T_x, T_y, T_z) > 0$,$F(G(T_x, T_y, T_z)) < \tau +$ $F\left(G(T_x, T_y, T_z)\right)(\max\{G(x, y, z), G(x, T_x, T_x), G(y, T_y, T_y),$ $G(z, T_z, T_z)\})$.

Then by (F1), we get

$$\begin{aligned}&(T_x, T_y, T_z)\\ &= \max\{G(x, y, z), G(x, T_x, T_x), G(y, T_y, T_y), G(z, T_z, T_z)\end{aligned}$$

$for\ all\ x, y, z \in X\ with\ (T_x, T_y, T_z) > 0.$

4) *Theorem 2.4*

Let (X, G) be a complete $G-$metric space. Let $: X \to X$ be an $F - weak$ contraction. If T or F is continuous, then T has a unique fixed point $x*$ in X and for every $x_0 \in X$, there is a sequence $\{T_n x_0\}$ in X that converges to x^*.

Proof:

Let $x_0 \in X$ be arbitrary. We define a sequence $\{xn\}$ in X given by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. if there exists $n_0 \in \mathbb{N}$ for which $x_{n_0+1} = x_{n_0}$ then $Tx_{n_0} = x_{n_0}$. this shows that x_{n_0} is a fixed point of T. Therefore, we assume that $x_{n+1} \neq x_n$ for every n $\in \mathbb{N} \cup \{0\}$

Let $p_n = (x_n, x_{n+1}, x_{n+2})$ for all $n \in \mathbb{N}$.

It follows from

a) The sequence is $\{x_n\}$ G-cauchy.
b) For every $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq k$ for each $n \in \mathbb{N}$,

$$F(p_n) = ((Tx_{n-1}, Tx_n, Tx_{n+1}))$$

$$\leq F(max\begin{Bmatrix}(x_{n-1}, x_n, x_{n+1}), (x_{n-1}, Tx_{n-1}, Tx_{n+1}) \\ (x_n, Tx_n, Tx_n), (x_{n+1}, Tx_{n+1}, Tx_{n+1})\end{Bmatrix} - \tau)$$

$$= (\max\{p_{n-1}, G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+2}, x_{n+2})\} - \tau)$$

$$= (p_{n-1}) - \tau$$

By successive application, we get for all $n \in \mathbb{N}$

$$(p_n) \leq (p_{n-1}) - (p_{n-2}) - 2 \leq \cdots \leq (p_0) - n\tau$$

Taking the limit as $n \to \infty$ in (1) we get $\lim_{n\to\infty} (p_n) = -\infty$ and then by of Let $F: \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying

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(F2) For every sequence $\{\alpha_n\}$ in $\mathbb{R}^+$ we have $\lim_{n\to\infty} = 0$ if and only if $\lim_{n\to\infty}(\alpha_n) = -\infty$, $\lim_{n\to\infty} = 0$

Now, by (F3) There exists a number $\in$ (0, 1) such that $\lim_{n\to\infty} F(\alpha) = 0$, there exists $k \in (0,1)$ such that $\lim_{n\to\infty}(p_n) = 0$, for all $n \in \mathbb{N}$.

$$p_n^k F(p_n) - p_n^k F(p_0) = p_n^k \left(F(p_n) - F(p_0)\right) \le n p_n^k \tau$$

Letting $n \to \infty$ in (28) and using (26) and (27) we have $\lim_{n\to\infty} = 0$

Then there exists a positive integer n_1 such that $np_n^k < 1$ for all $n \ge 1$. Consequently we show that $\{x_n\}$ is a Cauchy sequence. Now for all $m > n > 1$ we have

$$G(x_n, x_m, x_m) \le p_n + p_{n+1} + \cdots + p_m < \sum_{i=n}^{\infty} p_i \le \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}} \tag{31}$$

As $k \in (0,1)$, the series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{k}}}$ is convergent and therefore (31),$\{x_n\}$ is a Cauchy sequence in X. since X is complete, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$

Now we prove that x^* is a fixed point of T by considering the following two cases:

Case 1: If T is continuous.

We have
$(Tx^*, x^*, x^*) = \lim_{n\to\infty}(Tx_n, x_n, x_n) = \lim_{n\to\infty}(Tx_{n+1}, x_n, x_n) = 0$

This proves that x^* is a fixed point of T.

Case 2: If F is continuous.

We consider the following two subcases:

Subcase 1: There exists $n_0 \in \mathbb{N}$ such that $x_{n+1} \neq Te^*$ for all $n \ge n_0$

That is $(Tx_n, Tx^*, Tx^*) > 0$ for all $n \ge n_0$

It follows from

1. The sequence $\{x_n\}$ is G-cauchy.
2. For every $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $(x_n, x_m, x_m) < \varepsilon$, for all $n, m \ge k$ for each $n \in \mathbb{N}$,

$$\tau + ((x_{n+1}, Tx^*, Tx^*)) = \tau + ((x_n, Tx^*, Tx^*))$$
$$\le (max\{G(x_n, x^*, x^*), G(x_n, x_{n+1}, x_{n+1}), G(x^*, Tx^*, Tx^*)\})$$
$$= (max\{G(x_n, x^*, x^*), G(x_n, x_{n+1}, x_{n+1}), G(x^*, Tx^*, Tx^*)\})$$

If (x^*, Tx^*, Tx^*) , then using the fact

$$\lim_{n\to\infty} G(x_n, x^*, x^*) = \lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$$

There exists $n_1 \in \mathbb{N}$ such that for all $n \ge n_1$, we have

$$\{(x_n, x^*, x^*), (x_n, x_{n+1}, x_{n+1}), (x^*, Tx^*, Tx^*)\} = G(x^*, Tx^*, Tx^*)$$

Therefore we get

$$\tau + ((x_{n+1}, Tx^*, Tx^*)) \le (x^*, Tx^*, Tx^*), \forall n \ge \{n_0, n_1\} \tag{32}$$

Since F is continuous, taking limit as $n \to \infty$ in (32), we obtain $\tau + ((x_{n+1}, Tx^*, Tx^*)) \le (x^*, Tx^*, Tx^*)$ which is contradiction. Therefore $(x^*, Tx^*, Tx^*) = 0$ thus x^* is a fixed point of T

Subcase 2: There exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{k+1} = Tx^*$ for all $k \in \mathbb{N}$.

Then we have $x^* = \lim_{k\to\infty} x_{n+1_k} = \lim_{k\to\infty} Tx^* = Tx^*$.

This shows that x^* is a fixed point of T.

Combining above two cases, we get that T has a fixed point x^* in X. Now we show the uniqueness.

Let x^* and y^* be two fixed points of T. Suppose that $x^* \neq y^*$ then $Tx^* \neq Ty^*$

It follows from

a) The sequence is $\{x_n\}$ G-cauchy.
b) For every $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $(x_n, x_m, x_m) < \varepsilon$, for all $n, m \ge k$ for each $n \in \mathbb{N}$,

$$\tau + F(G(x^*, y^*, y^*)) = \tau + F(G(Tx^*, Ty^*, Ty^*))$$
$$\le (max\{G(x^*, y^*, y^*), G(x^*, Tx^*, Tx^*), G(x^*, Ty^*, Ty^*)\})$$
$$= (max\{G(x^*, y^*, y^*), G(x^*, x^*, x^*), G(x^*, y^*, y^*)\})$$
$= ((x^*, y^*, y^*))$ which is contradiction.

Thus, $(x^*, y^*, y^*) = 0$, that $x^* = y^*$.

Also, we have seen above that

$$x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty}(x_0)$$

5) Corollary 2.5

Let (X, G) be a complete G-metric space.

Let $T: X \to X$ satisfies:

$$\tau + F\left(G(T_x, T_y, T_z)\right) \le F\left(aG(x, y, z) + bG(x, T_x, T_x) + cG(y, T_y, T_y) + dG(z, T_z, T_z)\right) \tag{33}$$

For all $(x, y, z) \in X$ with $(T_x, T_y, T_z) > 0$ and $a, b, c \ge 0$ such that $a + b + c + d < 1$ if T or F is continuous, then T has a unique fixed point x^* in X and for every $x_0 \in X$there is a sequence $\{T^n x_0\}$ in X that converges to x^*.

Proof:

For all $(x, y, z) \in X$

$$aG(x, y, z) + bG(x, T_x, T_x) + cG(y, T_y, T_y) + dG(z, T_z, T_z)$$
$$\le (a + b + c + d)\, max\{G(x, y, z) + G(x, T_x, T_x) + G(y, T_y, T_y) + G(z, T_z, T_z)\}$$
$$< max\{G(x, y, z) + G(x, T_x, T_x) + G(y, T_y, T_y) + G(z, T_z, T_z)\}$$

Then by F is strictly increasing. That is,

$\alpha < \beta \Rightarrow F(\alpha) < F(\beta)$ for all $\alpha, \beta \in \mathbb{R}^+$.

1. The sequence $\{x_n\}$ is G-cauchy.
2. For every $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $(x_n, x_m, x_m) < \varepsilon$, for all $n, m \ge k$ for each $n \in \mathbb{N}$, is a consequence of (32)

Hence the corollary is proved.

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III. CONCLUSION

We conclude that this dissertation we deal about to achieve the unique common fixed-point theorem for two pair of weakly compatible maps in a complete metric space.

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