Analysis of Vector Space Related to Graph Theory

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Abstract— In our everyday life, we come across various problems where we have to look at them as structures of objects and some family of subsets of those objects. These problems may be represented pictorially with a set of dots called vertices and a set of edges connecting various pairs of dots. Such representations are called graphs. These problems involve arrangements of certain objects and explain the relationships between them. To study the arrangements, we view these objects as points in a plane and the relationships as lines joining them. Ideas given by various mathematicians to solve such problems gave birth to a branch of mathematics called graph theory.

Key words: Vector Space, Graphs, Abelion Group, Circuit, Cut Set, Disjoint Set

I. Introduction

Graph theory is a branch of mathematics with the relevance and application to at most all the Fields. This is one of the active research areas and grows rapidly.

The theory of graphs has a definite starting place in a paper published in 1736 by the swiss mathematician Leonhard Euler (1707 – 1783).

In 1847, while establishing the well-known. Kirchoff laws governing electrical networks, G. kirchoff initiated Algebraic graph Theory.

Cutsets play a vital role in communication and transportation problems. Suppose a few cities are connected by telephone lines. To find and strengthen the weak spots of the network, we concentrate on all cutsets with smallest number of edges.

Cutsets are very useful in the development of graph related vector spaces.

This dissertation entitled "vector spaces Related to Graphs" consists of four chapters.

II. Main Result

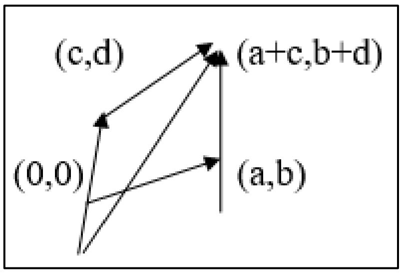
A. Definition: VECTOR SPACES

During the 19th century mathematicians and scientists were developing new tools for trying to maximize the way mathematics could be used to get insight into the concepts of location, velocity and force.

These ideas were the outgrowth of a long progression of mathematics set in motion by newton's spectacular synthesis of using mathematical tools for the benefit of physics. Newton was both a very great mathematician and a very great physicist.

We can represent velocities, which have a magnitude and direction, by line segments with arrows.

1) *Example*



Vector spaces arise is in the theory of equations of first degree in several variables.

2) *Example*

A linear equation in three variables is:

$$ax + by + cz = 0$$

Defines a plane in 3 – space which passes through the origin of 3-space, the point (0,0,0). It is easy to see all planes of the form pass through (0,0,0).

The geometry of 3– space is ∋ two different planes are either parallel or intersect in a line.

B. Definition: GRAPHS

Graphs are another class of structures widely studied by mathematicians and computer scientists. There are structures which consist of dots and lines.

Graphs have many theoretical as well as practical uses. Graphs and vector spaces seem like very different worlds, and in many respects they are, but in the next section we will see connections between the two.

Consider a graph G= (V,E,ψ_G). Let w_G=P(E), on this power set of E, we consider the operation 'symmetric difference of sets': which is also called the ring sum operation between sets. We denote it by. Now we show that this W_G with respect to this operation is an abelian group.

We know that $A \oplus B= (A-B)\cup(B-A)$.

1) w_G is closed under $\oplus$, since w_G contains all subsets of E including Ø.
2) for any $E_i \in w_G$, $E_i \oplus Ø = Ø \oplus E_i=E_i$. Hence
3) Ø is the identity element of w_G.
4) Since $E_i \oplus E_i= Ø$, each E_i in w_G has self inverse.
5) $\oplus$ is associative is clear.
6) $E_i \oplus E_j=E_j \oplus E_i$, $E_i,E_j \in w_G$

Hence $(w_G, \oplus)$ is an abelian group,

Now we consider the field $(Z_2,\oplus,\odot)$. we show that w_G is a vector space over this field Z_2, by defining a multiplication operation between w_G and Z_2 by,

$1 * E_i=E_i$ and $0 * E_i = Ø$ where $E_i \in w_G$. It can be easily seen that * satisfies other conditions of a vector space.

Hence $(w_G, \oplus, *)$ is a vector space over this field Z_2.

C. Theorem: 2.3

In the vector space of a graph, the circuit subspace and the cut-set subspace are orthogonal to each other.

1) Proof

By known theorem

"Every circuit has an even number of edges in common with any cut-set"

The number of edges common to a union of edges-disjoint circuits and a union of edges-disjoint cut-sets, this is also even can be shown as follows.

Let g_1 be a union of three edges-disjoint circuits Γ_1,Γ_2, and Γ_3 in a graph G, and g_2 be a union of two edges-disjoint cut-sets S_1 and S_2 in G.

Let the number of edges common to

Γ_1 and S_1 be 2a,
Γ_1 and S_2 be 2b,
Γ_2 and S_1 be 2c,
Γ_2 and S_2 be 2d,
Γ_3 and S_1 be 2e,
Γ_3 and S_2 be 2f,

Since there is no common between S_1 and S_2, or between Γ_1 and Γ_2 and Γ_3, the six sets of common edges enumerated above are the distinct (some may be empty).

Therefore, the number of edges common to g_1 and g_2 is

2a+2b+2c+2d+2e+2f, an even number.

This example can be extended to g_1 and g_2 to include the union of any finite of any finite numbers of edges-disjoint circuits and cut-sets, respectively.

From known theorem, the dot product of a circuit vector and a cut-sets vector is zero.

Hence every vector in each of these subspaces is orthogonal to every vector in the other.

D. Theorem: 2.4

The dimension of the vector space w_G over Z_2 is the number of edges in G.

Proof

We know that w_G is a vector space over Z_2.Consider that edge set E(G)={e1,e2,……eq}

Then {{e_1},{e_2},…….{eq}} is a basis for w_G. Since, let {$e\alpha_1$,$e\alpha_2$,……..$e\alpha_k$} be a member of w_G.

Where {α_1,α_2,……..α_k} $\subseteq${1,2,…………..q} of the preceding members.

Hence the dimension w_G is q, which is the number of edges in G.

E. Theorem: 2.5

The set of all cycles and unions of edge – disjoint cycles of a graph G, say w_c is a subspace of the vector space w_G of G.

Proof

Without loss of generality we assume that w_c consists of all edge – induced subgraphs of G. in which all vertices are of even degree.

It is enough if we proved that w_c is closed under the ring sum operation.

Let c_1,c_2 Є w_c.

Now we prove that c3 = c1 $\oplus$ c2 Є w_c.

Let V Є C_3.

This vertex should belong to at least one of the subgraphs c_1 and c_2. Let Xi denote the set of all edges incident on V in C_i, i=1,2,3. Thus $|X_i|$ is the degree of the vertex V in C_i. We note that $|X_1|$ and $|X_2|$ are even, one of them may be zero and $|X_3|$ is non – zero.

Since c3 = c1 $\oplus$ c2,we get x3 = x1 $\oplus$ X2, so, $|X_3|$= $|X_1|$ + $|X_2|$ -2 $|X_1 \cap X_2|$.

That is, degree of V in c_3 is even, Since V is arbitrary, c_3 Є w_c.

Hence the proof.

F. Theorem:2.6

The ring sum of any two cuts n a graph G is also a cut in G .

Proof:

Consider any two cuts s_1 = <V_1 ,V_2> and s_2 = <V_3,V_4> in a graph G.

Now, $V_1 U V_2$ = V= $V_3 U V_4$ and $V1 \cap V_2 = \emptyset = V_3 \cap V_4$.

Let A= $V_1 \cap V_3$.
B=$V_1 \cap V_4$
C=$V_2 \cap V_3$
And D=$V_2 \cap V_4$.

Clearly these sets are mutually disjoint.

Consider S_1=<AUB , CUD> = <A,C> U <A,D> U <B,C> U <B,D> And S_2 = < AUC,BUD>=<A,B> U <A,D> U <C,B> U <C,D>.

Hence $S_1 \oplus S_2$ = < AUD,BUC>.

Since (AUD)$\cap$(BUC) = Ø and (AUD) U (BUC) = V.

$S_1 \oplus S_2$ is a cut in G.

Hence the proof.

G. Theorem: 2.7

w_s, the set of all cutsets and unions of edge – disjoint cutsets in a graph G, is nothing but the set of all cuts in G.

Proof:

Since each cutset is also a cut and by the previous corollary union of any two edge-disjoint cuts in a graph G is also a cut in G, w_s is contained in the set of all cuts in G.

Also, since a cut in a connected graph G is a cutset or union of edge-disjoint cutsets of G, the set of all cuts in G is contained in w_s.

Hence, w_s is nothing but the set of all cuts in G

III. CONCLUSION

Motivated by the definition of Vector Spaces, Graphs. We have discussed above analysis of vector spaces and related to graph theory.

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