Generalized Conditions on S -Normal and S- Unitary Matrices

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Abstract— The concept of s-unitarily equivalent matrices is used to obtain some equivalent condition on s-Normal ana s-Unitary matrices. Further, Numerical examples are proved.
Key words: S-Normal, S-Unitary Matrices

I. INTRODUCTION

The concept of normal matrices plays an important role in the spectral theory of matrices and in the theory of generalized inverses. Unitary matrices have significant importance in quantum mechanics because they preserve norm [1-2]. Recently, Krishnamoorthy and Vijayakumar [3] obtained equivalent conditions on s-normal matrices. Also, some properties of secondary unitary matrices have been studied by Krishnamoorthy and Govindarasu [4].

II. MATHEMATIC FORMULATION

A. *Definition: 2.1*

A matrix $A \in C_{m\times m}$ is said to be secondary normal (s-normal) matrix
if $AA^\theta = A^\theta A$.

$$A = \begin{bmatrix} 1+3i & 0 \\ 0 & 2+4i \end{bmatrix}$$

$$A^s = \begin{bmatrix} 2+4i & 1 \\ 2 & 1+3i \end{bmatrix}$$

$$A^\theta = \bar{A} = \begin{bmatrix} 2-4i & 1 \\ 2 & 1-3i \end{bmatrix}$$

$$A A^\theta = \begin{bmatrix} 1+3i & 0 \\ 0 & 2+4i \end{bmatrix} \begin{bmatrix} 2-4i & 0 \\ 0 & 1-3i \end{bmatrix}$$

$$= \begin{bmatrix} 14+2i & 0 \\ 0 & 14-2i \end{bmatrix}$$

$$= A A^\theta$$

Thus A is s-normal matrix.

B. *Definition: 2.2*

A matrix $A \in C_{m\times m}$ is said to be s-unitary matrix,
if $A A^\theta = A^\theta A = I_n$.

Remark: 2.3

A matrix $A \in C_{m\times m}$ is s-unitary

$$A^{-1} = A^\theta \text{ or } A^{-1} = VA^*V$$

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ is an S- unitary matrix.}$$

$$A^\theta = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$AA^\theta = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Thus A is an S-Unitary matrix.

In the following we shall see the definition of secondary unitarily (s-unitarily) equivalent matrices.

C. *Definition: 2.4*

Let $A, B \in C_{m\times m}$. Then the matrix B is said to be s-unitarily equivalent to the matrix A, if there exists an s-unitary matrix U such that $B = UAU^\theta$

D. *Example: 2.5*

Also, $U^\theta U = I_2$
Therefore $UU^\theta = U^\theta U = I_2$ and U is s-unitary matrix.

$$\text{Let } A = \begin{bmatrix} 1+i & 2i \\ 3+2i & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 2+2i & 2+3i \\ -2+2i & -3+2i \end{bmatrix}$$

$$\text{For, } U = \begin{bmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}$$

$$UU^\theta = \begin{bmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix}$$

Also $UU^\theta = I_2$

$$\text{Now } U^\theta UA = \begin{bmatrix} -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1+i & 2i \\ 3+2i & 3 \end{bmatrix} \begin{bmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 2+2i & 2+3i \\ -2+2i & -3+2i \end{bmatrix}$$

Hence B is s-unitarily equivalent to A.

E. *Theorem: 2.6*

If $A \in C^{n\times n}$ is s-unitarily equivalent to a diagonal matrix D, then A is s-normal matrix.

Proof:

Since A is s-unitarily equivalent to a diagonal matrix D, then there exists a s-unitary matrix P, such that

$$A = PDP^\theta.$$

Now,

$$AA^\theta = (PDP^\theta)(PDP^\theta)^\theta$$
$$= PD(P^\theta P)D^\theta P^\theta$$
$$= P(DD^\theta)P^\theta$$
$$= P(D^\theta D)P^\theta$$
$$= (PD^\theta)I_n(DP^\theta)$$
$$= (PD^\theta P^\theta)(PDP^\theta)$$
$$= (PDP^\theta)^\theta(PDP^\theta)$$
$$= A^\theta A$$

Thus $AA^\theta = A^\theta A$ and A is s-normal.

F. *Theorem: 2.7*

A matrix $A \in C^{n\times n}$ is s-normal if and only if $A^{-1} A^\theta$ is s-unitary matrix.

Proof:

$A^{-1} A^\theta$ is s-unitary

$$(A^{-1} A^\theta)(A^{-1} A^\theta)^\theta = I_n$$
$$(A^{-1} A^\theta)(A(A^{-1})^\theta) = I_n$$
$$A^{-1}(A^\theta A)(A^\theta)^{-1} = I_n$$
$$A^\theta A = AA^\theta$$

A is s–normal.

Similarly, $A^{-1} A^\theta$ is s-unitary

$(A^{-1} A^{\theta})^{\theta} (A^{-1} A^{\theta}) = I_n$
$A(A^{-1})^{\theta} (A^{-1} A\theta) = I_n$
$A(A^{\theta})^{-1} (A^{-1} A^{\theta}) = I_n$
$A^{\theta}A = AA^{\theta}$

A is s-normal.

Hence it is proved.

Hence the theorem

G. *Theorem: 2.8*

Let $A \in C^{n\times n}$, Assume that A= VP where V is s-unitary and P is non-singular matrix and s-hermitian matrix such that if P^2 commutes with V, P also commutes with V.

Then the following are equivalent.

1) A is s-normal.
2) VP = PV.
3) AV=VA.
4) AP=PA.

H. *Proof:*

(i) => (ii):

Assume A is s-normal, we show that VP = PV.

A is s-normal => $AA^{\theta}= A^{\theta}A$.
=> $(VP) (VP)^{\theta} = (VP)^{\theta} (VP)$
(since A = VP)
=> $(VP) (P^{\theta}V^{\theta}) = (P^{\theta}V^{\theta}) (VP)$
=> $V(PP^{\theta})V = P(V^{\theta} V)P$
=> $VP^2V^{\theta} = P^2$ (since P is s-hermitian)
=> $VP^2(V^{\theta}V) = P^2V$ (since V is s-unitary)
=> $VP^2 = P^2V$
=> VP = PV

Thus (i) => (ii), (ii) => (iii):

Assume VP = PV, We shall show that AV = VA

Now AV = (VP) V (since A = VP)
= V (PV)
= V(VP) (since VP = PV)
= V(PV) , Which implies AV = VA

Thus (ii) => (iii), (iii) => (iv),

Assume AV = VA, show that AP = PA

Now,

AP = (VP) P
= (PV) P
= P (VP),

Which implies AP = PA

Thus (iii) => (iv), (iv) => (i),

Assume AP=PA, we show that A is s-normal.

Now,

AP = PA => (VP) P = P (VP).
=> (VP) P = (PV) P

Thus VP = PV implies is $P^{\theta}V^{\theta} = V^{\theta}P^{\theta}$

Also, $AA^{\theta} = (VP)(VP)^{\theta}$
$= V(PP^{\theta})V^{\theta}$
$= (VP)(P^{\theta}V^{\theta})$
$= (VP)(V^{\theta}P^{\theta})$
$= V(P^{\theta}V^{\theta}) P$
$= (VV^{\theta})(P^{\theta}P)$
$= V^{\theta} (VP)P^{\theta}$
$= V^{\theta}(PV)P^{\theta}$
$= V^{\theta}P^{\theta}(VP)$
$= (PV)^{\theta}(VP)$ $\{AA^{\theta} = (VP)^{\theta}(VP) = A^{\theta}A\}$

Therefore A is s-normal.

Hence the theorem is proved.

$A^{-1} A^{\theta}$ is s-unitary
$(A^{-1} A^{\theta}) (A^{-1} A^{\theta})^{\theta} = I_n$
$(A^{-1} A^{\theta}) (A(A^{-1})^{\theta}) = I_n$
$A^{-1}(A^{\theta}A) (A^{\theta})^{-1} = I_n$
$A^{\theta}A = AA^{\theta}$

A is s–normal.

Similarly, $A^{-1} A^{\theta}$ is s-unitary
$(A^{-1} A^{\theta})^{\theta} (A^{-1} A^{\theta}) = I_n$
$A(A^{-1})^{\theta} (A-1\ A\theta) = I_n$
$A(A^{\theta})^{-1} (A^{-1} A^{\theta}) = I_n$
$A^{\theta}A = AA^{\theta}$

A is s-normal.

Hence it is proved.

III. CONCLUSION

In matrix theory, the special types of matrices namely s-normal and s-unitary matrices are taken. We generalized some of the results are proved on s-normal and s-unitary matrices.

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