On Multiplicative K–Eccentric and Multiplicative K-Hyper Eccentric Indices of Hexagonal Chain

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Abstract— Let G be a connected graph with vertex set V(G) and edge set E(G). In this paper, we define the Multiplicative K-eccentric, and Multiplicative K-hyper eccentric indices.

Key words: Molecular Graph, Eccentricity, Multiplicative First and Second K-Eccentric Indices, and Multiplicative First and Second K-Hyper Eccentric Indices

I. INTRODUCTION

All graphs considered in this paper are finite, connected, undirected without loops and multiple edges. Any undefined term here may be found in Kulli [2].

Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = m$. The eccentricity $e_G(v)$ of a vertex v is the distance of any vertex farthest from v. Let $e = uv \in E(G)$. Let $e_{L(G)}(e)$ denote the eccentricity of an edge e in L(G), where L(G) is the line graph of G. The vertices and edges of a graph are called the elements of G.

The line graph of an undirected graph G is another graph L(G) that represents the adjacencies between edges of G. A Line graph of a simple graph is obtained by associating a vertex with each edge of the graph and connecting two vertices with an edge if the corresponding edges of G have a vertex in common.

In [3], Kulli introduced the first and second K Banhatti indices to take account of the contributions of pairs of incident elements. And it was defined as

$$B_1(G) = \sum_{ue}[d_G(u) + d_G(e)] \qquad (1)$$
$$B_2(G) = \sum_{ue}[d_G(u)d_G(e)] \qquad (2)$$

In [4], Kulli introduced the first and second K Hyper Banhatti indices and it was defined as

$$HB_1(G) = \sum_{ue}[d_G(u) + d_G(e)]^2 \qquad (3)$$
$$HB_2(G) = \sum_{ue}[d_G(u)d_G(e)]^2 \qquad (4)$$

In [5,6] he introduced the Multiplicative first and second K Banhatti indices and it was defined as

$$B\prod_1(G) = \prod_{ue}[d_G(u) + d_G(e)] \qquad (5)$$
$$B\prod_2(G) = \prod_{ue}[d_G(u)d_G(e)] \qquad (6)$$

Similarly in [7], Kulli introduced the Multiplicative first and second K Hyper Banhatti indices and it was defined as

$$HB\prod_1(G) = \prod_{ue}[d_G(u) + d_G(e)]^2 \qquad (7)$$
$$HB\prod_2(G) = \prod_{ue}[d_G(u)d_G(e)]^2 \qquad (8)$$

In [8], we define Multiplicative First and second K-Eccentric indices are defined as,

$$B\prod_1 E(G) = \prod_{ue}\left[e_G(u) + e_{L(G)}(e)\right]. \qquad (9)$$
$$B\prod_2 E(G) = \prod_{ue}\left[e_G(u)e_{L(G)}(e)\right] \qquad (10)$$

The Multiplicative First and second K Hyper - Eccentric indices are defined as,

$$HB\prod_1 E(G) = \prod_{ue}\left[e_G(u) + e_{L(G)}(e)\right]^2 \qquad (11)$$
$$HB\prod_2 E(G) = \prod_{ue}\left[e_G(u)e_{L(G)}(e)\right]^2 \qquad (12)$$

Where in all the cases ue means that the vertex u and edge e are incident in G and $e_{L(G)}(e)$ is the eccentricity of e in the line graph L(G) of G.

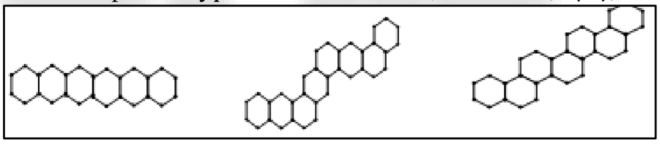
II. HEXAGONAL CHAIN

A Hexagonal system is a 2-connected plane graph whose every interior face is bounded by a regular hexagon of unit length 1. Some examples of hexagonal chains can be found in Fig. 1.

It is easy to see that any hexagonal chains H_{n+1} with $n + 1$ hexagons can be obtained from a hexagonal chain Hn with n hexagons by attaching it to a new hexagon. Based on this fact, a hexagonal chain can be constructed inductively. There are three types of fusion for attaching a new hexagon h_{n+1} to a hexagonal chain H_n with n hexagons $h_1, h_2, h_3, \ldots . h_n$:

- If h_{n+1} is on the line l, it is called $\propto -$type fusing;
- If h_{n+1} is on the left-hand side of l, it is called $\beta -$ type fusing;
- If h_{n+1} is on the right-hand side of l, it is called $\gamma -$ type fusing

Where l is the direct line from the center of h_{n-1} to the center of h_n. Any hexagonal chain $H_n (n \geq 2)$ can be obtained from H_2 by a stepwise fusion of new hexagons, and at each step a $\theta -$ type fusion is selected, Where $\in \{\alpha, \beta, \gamma\}$.



(i) A linear chain L6 (ii) A hexagonal chain H8 (iii) A zia-zag chain Z7

Fig. 1: Single Hexagonal chains with different types of fusions.

III. MULTIPLICATIVE FIRST AND SECOND K HYPER ECCENTRIC INDICES OF HEXAGONAL SYSTEM

Let G be a graph with vertex set V(G) and edge set E(G). The eccentricities of $u, v \in V(G)$ are denoted by e_u, e_v. For $e = uv \in E(G)$, denote the eccentricities of the end vertices of e by (e_u, e_v). In the following figures red color indicates eccentricities of the end vertices in G and black color indicates eccentricities of the end vertices in L(G).

A. *Theorem 3.1*

Let $L_4, H_4, and Z_4,$ are the Linear chain with $\propto -type$ fusing, hexagonal chain with $\beta - type$ fusing, and Zig-zag chain with $\gamma - type$ fusing respectively, then

$$\text{(i)(a)} B\prod_1 E(L_4) = \begin{cases} 3.7224 \times e^{47}, \propto -\text{type fusing} \\ 1.5213 \times e^{45}, \beta - \text{type fusing} \\ 1.5213 \times e^{45}, \gamma - \text{type fusing} \end{cases}$$

$$(b)B\prod_2 E(L_4) = \begin{cases} 7.5826\times e^{69}, \propto -\text{type fusing}\\ 2.5546\times e^{61}, \beta-\text{type fusing}\\ 2.5546\times e^{61}, \gamma-\text{type fusing}\end{cases}$$

$$(c)HB\prod_1 E(L_4) = \begin{cases} 2.2535\times e^{110}, \propto -\text{type fusing}\\ 4.6778\times e^{86}, \beta-\text{type fusing}\\ 4.6778\times e^{86}, \gamma-\text{type fusing}\end{cases}$$

$$(d)HB\prod_2 E(L_4) = \begin{cases} 4.2982\times e^{139}, \propto -\text{type fusing}\\ 5.0747\times e^{158}, \beta-\text{type fusing}\\ 5.0747\times e^{158}, \gamma-\text{type fusing}\end{cases}$$

$$(ii)(a)B\prod_1 E(H_4) = B\prod_1 E(Z_4) = \begin{cases} 3.0449\times e^{25}, \propto -\text{type fusing}\\ 2.6980\times e^{90}, \beta-\text{type fusing}\\ 6.4313\times e^{94}, \gamma-\text{type fusing}\end{cases}$$

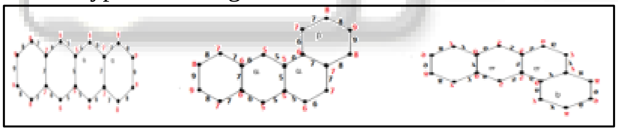
$$(b)B\prod_2 E(H_4) = B\prod_2 E(Z_4) = \begin{cases} 3.0623\times e^{36}, \propto -\text{type fusing}\\ 9.0362\times e^{131}, \beta-\text{type fusing}\\ 4.4307\times e^{140}, \gamma-\text{type fusing}\end{cases}$$

$$(c)HB\prod_1 E(H_4) = HB\prod_1 E(Z_4) = \begin{cases} 9.2334\times e^{50}, \propto -\text{type fusing}\\ 5.0353\times e^{58}, \beta-\text{type fusing}\\ 2.5354\times e^{63}, \gamma-\text{type fusing}\end{cases}$$

$$(d)HB\prod_2 E(H_4) = HB\prod_2 E(Z_4) = \begin{cases} 9.3774\times e^{72}, \propto -\text{type fusing}\\ 3.0070\times e^{69}, \beta-\text{type fusing}\\ 2.9386\times e^{95}, \gamma-\text{type fusing}\end{cases}$$

1) Proof

Consider the Linear chainL_4 in a line with $\propto -$type, $\beta-$type and $\gamma-$type fusing and which is denoted as $L_4(\alpha,\alpha)$, $L_4(\alpha,\beta)$ and $L_4(\alpha,\gamma)$. Fig 5. Shows L_4, with different types of fusing



$L_4(\alpha,\alpha)$ $L_4(\alpha,\beta)$ $L_4(\alpha,\gamma)$

Fig. 2: Linear chainL_4 with α,β,γ type fusing

Let V_1 be the vertex set and E_1 be the edge set in $L_4(\alpha,\alpha)$, then $|V_1| = 18$ and$|E_1| = 21$. Also the number of edges with eccentricities of end vertices of G and L(G) are given as follows:

Let$E_1{}^1 = \{e\in E(G)\ /\ e = uv\ , u, v\in V_1(G),\ e_G(u) = 5, e_G(v) = 5\}$

For $e\in E_1{}^1, e_{L(G)}(e) = 5$ and $|E_1{}^1| = 1$

$E_1{}^2 = \{e\in E(G)\ /\ e = uv\ , u, v\in V_1(G),\ e_G(u) = 5, e_G(v) = 6\}$

For $e\in E_1{}^2, e_{L(G)}(e) = 5$ and $|E_1{}^2| = 4$ ...

$E_1{}^7 = \{e\in E(G)\ /\ e = uv\ , u, v\in V_1(G), e_G(u) = 9, e_G(v) = 9\}$

For $e\in E_1{}^7, e_{L(G)}(e) = 9$ and $|E_1{}^7| = 22`1$

Hence

$$(i)(a)B\prod_1 E(L_4(\alpha,\alpha)) = \prod_{ue}[e_G(u) + e_{L(G)}(e)]$$

$$B\prod_1 E(G) = \prod_{ue}[e_{L_4(\alpha,\alpha)}(u) + e_{L(L_4(\alpha,\alpha))}(e)]$$

$$= \prod_{e=uv\in E_1(G)}[e_G(u) + e_{L(G)}(e)][e_G(v) + e_{L(G)}(e)]$$

$$= \prod_{e\in E_1{}^1(G)}[e_G(u) + e_{L(G)}(e)]\,[e_G(u) + e_{L(G)}(e)]$$

$$\times \prod_{e\in E_1{}^2(G)}[e_G(u) + e_{L(G)}(e)]\,[e_G(u) + e_{L(G)}(e)]\times\dots$$

$$\times \prod_{e\in E_1{}^7(G)}[e_G(u) + e_{L(G)}(e)]\,[e_G(u) + e_{L(G)}(e)]$$

$$= (5+5)^1(5+5)^1\times(5+5)^4(6+5)^4\times(7+6)^4(6+6)^4\times(7+7)^2(7+7)^2\times(7+7)^4(8+7)^4\times(8+8)^4(8+9)^4\times(9+9)^2(9+9)^2 = 3.7224e^{47}$$

$$(b)B\prod_2 E(L_4(\alpha,\alpha)) = \prod_{ue}[e_{L_4(\alpha,\alpha)}(u)e_{L(L_4(\alpha,\alpha))}(e)] = \prod\nolimits_{e=uv\in E_1(G)}[e_G(u)e_{L(G)}(e)][e_G(v)e_{L(G)}(e)] =$$

$$\prod_{e\in E_1{}^1(G)}[e_G(u)e_{L(G)}(e)]\,[e_G(u)e_{L(G)}(e)]$$

$$\times \prod_{e\in E_1{}^2(G)}[e_G(u)e_{L(G)}(e)]\,[e_G(u) + e_{L(G)}(e)]\times\dots\times$$

$$\prod_{e\in E_1{}^7(G)}[e_G(u)e_{L(G)}(e)]\,[e_G(u)e_{L(G)}(e)]$$

$$= (5\times5)^1(5\times5)^1\times(5\times5)^4(6\times5)^4\times(7\times6)^4(6\times6)^4\times(7\times7)^2(7\times7)^2\times(7\times7)^4(8\times7)^4\times(8\times8)^4(8\times9)^4\times(9\times9)^2(9\times9)^2 = 7.58263\times e^{69}$$

$$(c)HB\prod_1 E(L_4(\alpha,\alpha)) = \prod_{ue}[e_{L_4(\alpha,\alpha)}(u) + e_{L(L_4(\alpha,\alpha))}(e)]$$

$$= \prod_{e=uv\in E_1(G)}[e_G(u) + e_{L(G)}(e)]^2[e_G(v) + e_{L(G)}(e)]^2 =$$

$$\prod_{e\in E_1{}^1(G)}[e_G(u) + e_{L(G)}(e)]^2\,[e_G(u) + e_{L(G)}(e)]^2$$

$$\times \prod_{e\in E_1{}^2(G)}[e_G(u) + e_{L(G)}(e)]^2\,[e_G(u) + e_{L(G)}(e)]^2\times\dots$$

$$\times \prod_{e\in E_1{}^6(G)}[e_G(u) + e_{L(G)}(e)]^2\ 2$$

$$\times \prod_{e\in E_1{}^7(G)}[e_G(u) + e_{L(G)}(e)]^2\,[e_G(u) + e_{L(G)}(e)]^2$$

$$= (5+5)^2(5+5)^2\times(5+5)^8(6+5)^8\times(7+6)^8(6+6)^8\times(7+7)^8(7+7)^8\times(7+7)^8(8+7)^8\times(8+8)^8(8+9)^8\times(9+9)^8(9+9)^8 = 2.2535\times e^{110}$$

$$(d)HB\prod_2 E(L_4(\alpha,\alpha)) = \prod_{ue}[e_{L_4(\alpha,\alpha)}(u)e_{L(L_4(\alpha,\alpha))}(e)]^2$$

$$= \prod_{e\in E_1{}^1(G)}[e_G(u)e_{L(G)}(e)]^2\,[e_G(u)e_{L(G)}(e)]^2$$

$$\times \prod_{e\in E_1{}^2(G)}[e_G(u)e_{L(G)}(e)]^2\,[e_G(u) + e_{L(G)}(e)]^2\times\dots$$

$$\times \prod_{e\in E_1{}^7(G)}[e_G(u)e_{L(G)}(e)]^2\,[e_G(u)e_{L(G)}(e)]^2$$

$$= (5\times5)^2(5\times5)^2\times(5\times5)^8(6\times5)^8\times(7\times6)^8(6\times6)^8\times(7\times7)^4(7\times7)^4\times(7\times7)^8(8\times7)^8\times(8\times8)^8(8\times9)^8\times(9\times9)^4(9\times9)^4 = 4.2982\times e^{139}$$

Let V_2 be the vertex set and E_2 be the edge set in $L_4(\alpha,\beta)$, then $|V_2| = 18$ and $n|E_2| = 21$. The number of edges with eccentricities of end vertices of G and L(G) are given as follows:

Let$E_2{}^1 = \{e\in E(G)\ /\ e = uv\ , u, v\in V_2(G),\ e_G(u) = 5, e_G(v) = 5\}$

For $e\in E_2{}^1, e_{L(G)}(e) = 5$ and $|E_2{}^1| = 3$

$E_2{}^2 = \{e\in E(G)\ /\ e = uv, u, v\in V_2(G), \}e_G(u) = 5, e_G(v) = 6$

For $e\in E_2{}^2, e_{L(G)}(e) = 5$ and $|E_2{}^2| = 2$ ...

$E_2{}^8 = \{e\in E(G)/e = uv, u, v\in V_2(G)\ e_G(u) = 8, e_G(v) = 9\}$

For $e \in E_2{}^8, e_{L(G)}(e) = 9$ and $|E_2{}^8| = 3$

Hence

$$(i)(a)\ B\prod_1 E(L_4(\alpha,\beta)) = \prod_{ue}[e_G(u) + e_{L(G)}(e)]$$

$$B\prod_1 E(G) = \prod_{ue}[e_{L_4(\alpha,\beta)}(u) + e_{L(L_4(\alpha,\beta))}(e)]$$

$$= \prod_{e=uv\in E_2(G)} [e_G(u) + e_{L(G)}(e)][e_G(v) + e_{L(G)}(e)]$$

$$= \prod_{e\in E_2{}^1(G)} [e_G(u) + e_{L(G)}(e)]\,[e_G(u) + e_{L(G)}(e)]$$

$$\times \prod_{e\in E_2{}^2(G)} [e_G(u) + e_{L(G)}(e)]\,[e_G(u) + e_{L(G)}(e)] \times \dots$$

$$\times \prod_{e\in E_2{}^8(G)} [e_G(u) + e_{L(G)}(e)]\,[e_G(u) + e_{L(G)}(e)] =$$

$$[(5+5)^3 \times (5+5)^3] \times [(5+5)^2 \times (6+5)^2] \times [(5+5)^2 \times (6+6)^2] \times [(7+7)^2 \times (8+7)^2] \times [(7+8)^3 \times (8+8)^3] \times [(8+9)^3 + (9+9)^3] \times [(7+6)^4 \times (7+7)^4] \times [(6+6)^2 \times (7+6)^2] = 1.5213 \times e^{45}$$

$$(b)B\prod_2 E(G) = \prod_{ue}[e_{L_4(\alpha,\beta)}(u)e_{L(L_4(\alpha,\beta))}(e)]$$

$$= \prod_{e=uv\in E_2(G)} [e_G(u)e_{L(G)}(e)][e_G(v)e_{L(G)}(e)] =$$

$$\prod_{e\in E_2{}^1(G)} [e_G(u)e_{L(G)}(e)]\,[e_G(u)e_{L(G)}(e)]$$

$$\times \prod_{e\in E_2{}^2(G)} [e_G(u)e_{L(G)}(e)]\,[e_G(u)e_{L(G)}(e)] \times \dots \times$$

$$\prod_{e\in E_2{}^8(G)} [e_G(u)e_{L(G)}(e)]\,[e_G(u)e_{L(G)}(e)]$$

$$= [(5\times5)^3 \times (5\times5)^3] \times [(5\times5)^2 \times (6\times5)^2] \times [(5\times5)^2 \times (6\times6)^2] \times [(7\times6)^4 \times (7\times7)^4] \times [(6\times6)^2 \times (7\times6)^2] \times [(7\times7)^2 \times (8\times7)^2] \times [(7\times8)^3 \times (8\times8)^3] \times [(8\times9)^3 + (9\times9)^3] = 2.5546 \times e^{61}$$

$$(c)\ HB\prod_1 E(L_4(\alpha,\beta)) = \prod_{ue}[e_G(u) + e_{L(G)}(e)]^2 =$$

$$HB\prod_1 E(G) = \prod_{ue}[e_{L_4(\alpha,\beta)}(u) + e_{L(L_4(\alpha,\beta))}(e)]^2$$

$$= \prod_{e=uv\in E_2(G)} [e_G(u) + e_{L(G)}(e)]^2[e_G(v) + e_{L(G)}(e)]^2$$

$$= \prod_{e\in E_2{}^1(G)} [e_G(u) + e_{L(G)}(e)]^2\,[e_G(u) + e_{L(G)}(e)]^2 \times$$

$$\prod_{e\in E_2{}^2(G)} [e_G(u) + e_{L(G)}(e)]^2\,[e_G(u) + e_{L(G)}(e)]^2 \times \dots$$

$$\times \prod_{e\in E_2{}^8(G)} [e_G(u) + e_{L(G)}(e)]^2\,[e_G(v) + e_{L(G)}(e)]^2$$

$$= [(5+5)^6 \times (5+5)^6] \times [(5+5)^4 \times (6+5)^4] \times [(5+5)^4 \times (6+6)^4] \times [(7+7)^4 \times (8+7)^4] \times [(7+8)^6 \times (8+8)^6] \times [(8+9)^6 + (9+9)^6] \times [(7+6)^8 \times (7+7)^8] \times [(6+6)^4 \times (7+6)^4] = 4.6778 \times e^{86}$$

$$(d)HB\prod_2 E(G) = \prod_{ue}[e_{L_4(\alpha,\beta)}(u)e_{L(L_4(\alpha,\beta))}(e)]^2$$

$$= \prod_{e=uv\in E_2(G)} [e_G(u)e_{L(G)}(e)]^2[e_G(v)e_{L(G)}(e)]^2 =$$

$$\prod_{e\in E_2{}^1(G)} [e_G(u)e_{L(G)}(e)]^2\,[e_G(v)e_{L(G)}(e)]^2$$

$$\times \prod_{e\in E_2{}^2(G)} [e_G(u)e_{L(G)}(e)]^2\,[e_G(v)e_{L(G)}(e)]^2 \times \dots$$

$$\times \prod_{e\in E_2{}^8(G)} [e_G(u)e_{L(G)}(e)]^2\,[e_G(v)e_{L(G)}(e)]^2$$

$$= [(5\times5)^6 \times (5\times5)^6] \times [(5\times5)^4 \times (6\times5)^4] \times [(5\times5)^4 \times (6\times6)^{24}] \times [(7\times7)^4 \times (8\times7)^4] \times [(7\times8)^6 \times (8\times8)^6] \times [(8\times9)^6 + (9\times9)^6] \times [(7\times6)^8 \times (7\times7)^8] \times [(6\times6)^4 \times (7\times6)^4] = 5.0747 \times e^{158}$$

Similarly, let V_3 be the vertex set and E_3 be the edge set in $L_4(\alpha,\gamma)$, then $|V_3| = 18$ and $|E_3| = 21$. Also $L_4(\alpha,\gamma) = L_4(\alpha,\beta)$ since they are isomorphic .Hence we have

$(i)(a)\ B_1E(L_4(\alpha,\gamma)) = B_1E(L_4(\alpha,\beta)) = 1.5213 \times e^{45}$

And $(b)B_2E(L_4(\alpha,\gamma)) = B_2E(L_4(\alpha,\beta)) = 2.5546 \times e^{61}$

$(c)\ HB_1E(L_4(\alpha,\gamma)) = B_1E(L_4(\alpha,\beta)) = 4.6778 \times e^{86}$

$(d)HB_2E(L_4(\alpha,\gamma)) = B_2E(L_4(\alpha,\beta)) = 5.0747 \times e^{158}$

(ii) Consider the Hexogonal chainH_4, in a line with $\propto -$type, $\beta -$ type and $\gamma -$ type fusing and which is denoted as $H_4(\beta,\alpha)$, $H_4(\beta,\beta)$ and$H_4(\beta,\gamma)$ .Fig 3. Shows H_4 with different types of fusing

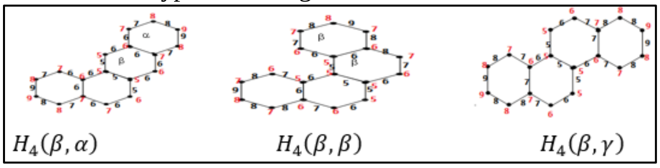


Fig. 3: Hexogonal chainH_4, with α,β,γ type fusing

Let V_4 be the vertex set and E_4 be the edge set in $H_4(\beta,\alpha)$, then $|V_4| = 18$ and$|E_4| = 21$. Also the number of edges with eccentricities of end vertices are given as follows:

Let $E_4{}^1 = \{e \in E(G)\ /\ e = uv\ , u,v \in V_4(G),\ e_G(u) = 5, e_G(v) = 5\}$

For $e \in E_4{}^1, e_{L(G)}(e) = 5$ and $|E_4{}^1| = 2$

$E_4{}^2 = \{e \in E(G)\ /\ e = uv\ , u,v \in V_4(G),\ e_G(u) = 5, e_G(v) = 6\}$

For $e \in E_4{}^2, e_{L(G)}(e) = 5$ and $|E_4{}^2| = 2$ ...

$E_4{}^8 = \{e \in E(G)\ /\ e = uv\ , u,v \in V_4(G), e_G(u) = 8, e_G(v) = 9\}$

For $e \in E_4{}^8, e_{L(G)}(e) = 9$ and $|E_4{}^8| = 2$

Hence

$$(ii)(a)\ B\prod_1 E(H_4(\beta,\alpha)) = \prod_{ue}[e_G(u) + e_{L(G)}(e)]$$

$$B\prod_1 E(G) = \prod_{ue}[e_{H_4(\beta,\alpha)}(u) + e_{L(H_4(\beta,\alpha))}(e)]$$

$$= \prod_{e=uv\in E_4(G)} [e_G(u) + e_{L(G)}(e)][e_G(v) + e_{L(G)}(e)]$$

$$= \prod_{e\in E_4{}^1(G)} [e_G(u) + e_{L(G)}(e)]\,[e_G(u) + e_{L(G)}(e)]$$

$$\times \prod_{e\in E_4{}^2(G)} [e_G(u) + e_{L(G)}(e)]\,[e_G(u) + e_{L(G)}(e)] \times \dots$$

$$\times \prod_{e\in E_4{}^7(G)} [e_G(u) + e_{L(G)}(e)]\,[e_G(u) + e_{L(G)}(e)]$$

$$\times \prod_{e\in E_4{}^8(G)} [e_G(u) + e_{L(G)}(e)]\,[e_G(u) + e_{L(G)}(e)] =$$

$$[(5+5)^2 \times (5+5)^2] \times [(5+5)^2 \times (6+5)^2] \times [(5+6)^2 \times (6+6)^2] \times [(7+7)^4 \times (8+7)^4] \times [(8+8)^2 \times (8+9)^2] \times [(8+9)^2 \times (9+9)^2] \times [(7+6)^6 \times (6+6)^6] + [(7+6) \times (7+6)] = 3.0449 \times e^{25}$$

$$(b)B\prod_2 E(H_4(\beta,\alpha)) = \prod_{ue}[e_G(u)e_{L(G)}(e)]$$

$$B\prod_2 E(G) = \prod_{ue}[e_{H_4(\beta,\alpha)}(u)e_{L(H_4(\beta,\alpha))}(e)]$$

$$= \prod_{e=uv\in E_4(G)} [e_G(u)e_{L(G)}(e)][e_G(v)e_{L(G)}(e)] =$$

$$\prod_{e\in E_4{}^1(G)} [e_G(u)e_{L(G)}(e)]\,[e_G(u)e_{L(G)}(e)]$$

$$\times \prod_{e\in E_4{}^2(G)} [e_G(u)e_{L(G)}(e)]\,[e_G(u)e_{L(G)}(e)] \times \ldots$$
$$\times \prod_{e\in E_4{}^8(G)} [e_G(u)e_{L(G)}(e)]\,[e_G(u)e_{L(G)}(e)]$$
$$= [(5\times5)^2\times(5\times5)^2]\times[(5\times5)^2\times(6\times5)^2] \times[(5\times6)^2\times(6\times6)^2]\times[(7\times7)^4+(8\times7)^4] \times[(7\times6)^6\times(6\times6)^6]+[(7\times6)\times(7\times6)] \times[(8\times8)^2\times(8\times9)^2]\times[(8+9)^2\times(9\times9)^2] = 3.0623\times e^{36}$$

(c) $$HB\prod_1 E(H_4(\beta,\alpha)) = \prod_{ue}[e_G(u)+e_{L(G)}(e)]^2$$
$$HB\prod_1 E(G) = \prod_{ue}[e_{H_4(\beta,\alpha)}(u)+e_{L(H_4(\beta,\alpha))}(e)]^2$$
$$= \prod_{e=uv\in E_4(G)} [e_G(u)+e_{L(G)}(e)]^2[e_G(v)+e_{L(G)}(e)]^2 =$$
$$\prod\nolimits_{e\in E_4{}^1(G)}[e_G(u)+e_{L(G)}(e)]^2\,[e_G(u)+e_{L(G)}(e)]^2\times \prod\nolimits_{e\in E_4{}^2(G)}[e_G(u)+e_{L(G)}(e)]^2\,[e_G(u)+e_{L(G)}(e)]^2\times\ldots\times \prod\nolimits_{e\in E_4{}^8(G)}[e_G(u)+e_{L(G)}(e)]^2\,[e_G(u)+e_{L(G)}(e)]^2$$
$$= [(5+5)^4\times(5+5)^4]\times[(5+5)^4\times(6+5)^4] \times[(5+6)^4\times(6+6)^4]\times[(7+7)^8\times(8+7)^8] \times[(7+6)^{12}\times(6+6)^{12}]+[(7+6)^2\times(7+6)^2] \times[(8+8)^4\times(8+9)^4]\times[(8+9)^4\times(9+9)^4] = 9.2334\times e^{50}$$

(d) $$HB\prod_2 E(H_4(\beta,\alpha)) = \prod_{ue}[e_G(u)e_{L(G)}(e)]^2$$
$$HB\prod_2 E(G) = \prod_{ue}[e_{H_4(\beta,\alpha)}(u)e_{L(H_4(\beta,\alpha))}(e)]^2$$
$$= \prod_{e=uv\in E_4(G)} [e_G(u)e_{L(G)}(e)]^2[e_G(v)e_{L(G)}(e)]^2 =$$
$$\prod_{e\in E_4{}^1(G)} [e_G(u)e_{L(G)}(e)]^2\,[e_G(u)e_{L(G)}(e)]^2$$
$$\times \prod_{e\in E_4{}^2(G)} [e_G(u)e_{L(G)}(e)]^2\,[e_G(u)e_{L(G)}(e)]^2 \times\ldots$$
$$\times \prod_{e\in E_4{}^8(G)} [e_G(u)e_{L(G)}(e)]^2\,[e_G(u)e_{L(G)}(e)]^2$$
$$= [(5\times5)^4\times(5\times5)^4]\times[(5\times5)^4\times(6\times5)^4] \times[(5\times6)^4\times(6\times6)^4]\times[(7\times7)^8+(8\times7)^8] \times[(7\times6)^{12}\times(6\times6)^{12}]+[(7\times6)^2\times(7\times6)^2] \times[(8\times8)^4\times(8\times9)^4]\times[(8+9)^4\times(9\times9)^4] = 9.3774\times e^{72}$$

Also let V_5 be the vertex set and E_5 be the edge set in $H_4(\beta,\beta)$, then $|V_5| = 18$ and $|E_5| = 21$. The number of edges with eccentricities of end vertices are given as follows:

Let $E_5{}^1 = \{e\in E(G)\ /\ e = uv\ , u,v\in V_5(G),\ e_G(u) = 5, e_G(v) = 5\}$

For $e\in E_5{}^1, e_{L(G)}(e) = 5$ and $|E_5{}^1| = 3$

$E_5{}^2 = \{e\in E(G)\ /\ e = uv\ , u,v\in V_5(G),\ e_G(u) = 5, e_G(v) = 6\}$

For $e\in E_5{}^2, e_{L(G)}(e) = 6$ and $|E_5{}^2| = 5 \ldots$

$E_5{}^7 = \{e\in E(G)\ /\ e = uv\ , u,v\in V_5(G), e_G(u) = 7, e_G(v) = 8\}$

For $e\in E_5{}^7, e_{L(G)}(e) = 9$ and $|E_5{}^7| = 2$

Hence

(ii)(a) $$B\prod_1 E(H_4(\beta,\beta)) = \prod_{ue}[e_G(u)+e_{L(G)}(e)]$$
$$B\prod_1 E(G) = \prod_{ue}[e_{H_4(\beta,\beta)}(u)+e_{L(H_4(\beta,\beta))}(e)]$$
$$= \prod_{e=uv\in E_5(G)} [e_G(u)+e_{L(G)}(e)][e_G(v)+e_{L(G)}(e)] =$$
$$\prod_{e\in E_5{}^1(G)} [e_G(u)+e_{L(G)}(e)]\,[e_G(u)+e_{L(G)}(e)]$$
$$\times \prod_{e\in E_5{}^2(G)} [e_G(u)+e_{L(G)}(e)]\,[e_G(u)+e_{L(G)}(e)]\times\ldots$$
$$\times \prod_{e\in E_5{}^7(G)} [e_G(u)+e_{L(G)}(e)]\,[e_G(u)+e_{L(G)}(e)]$$
$$= [(5+5)^3\times(5+5)^3]\times[(5+7)^2\times(6+7)^2] \times[(5+6)^2\times(6+6)^2]\times[(7+7)^3\times(6+7)^3] \times[(8+6)^4\times(7+8)^4]\times[(7+8)^4\times(8+8)^4] \times[(7+9)^2\times(8+9)^2] = 2.6980\times e^{90}$$

(b) $$B\prod_2 E(H_4(\beta,\beta)) = \sum_{ue}[e_{H_4(\beta,\beta)}(u)\times e_{L(H_4(\beta,\beta))}(e)]$$
$$= \sum_{e=uv\in E_5(G)} [e_G(u)+e_{L(G)}(e)\times e_G(u)+e_{L(G)}(e) =]$$
$$\prod_{e\in E_5{}^1(G)} [e_G(u)e_{L(G)}(e)]\,[e_G(u)e_{L(G)}(e)]$$
$$\times \prod_{e\in E_5{}^2(G)} [e_G(u)e_{L(G)}(e)]\,[e_G(u)+e_{L(G)}(e)]\times\ldots$$
$$\times \prod_{e\in E_5{}^7(G)} [e_G(u)e_{L(G)}(e)]\,[e_G(u)e_{L(G)}(e)]$$
$$= [(5\times5)^3\times(5\times5)^3]\times[(5\times7)^2(6\times7)^2] \times[(5\times6)^2\times(6\times6)^2]\times[(7\times7)^3\times(6\times7)^3] \times[(8\times6)^4\times(7\times8)^4]\times[(7\times8)^4\times(8\times8)^4]\times [(7\times9)^2\times(8\times9)^2] = 9.0362\times e^{131}$$

(c) $$HB\prod_1 E(H_4(\beta,\beta)) = \prod_{ue}[e_G(u)+e_{L(G)}(e)]^2$$
$$HB\prod_1 E(G) = \prod_{ue}[e_{H_4(\beta,\beta)}(u)+e_{L(H_4(\beta,\beta))}(e)]^2$$
$$= \prod_{e=uv\in E_5(G)} [e_G(u)+e_{L(G)}(e)]^2[e_G(v)+e_{L(G)}(e)]^2 =$$
$$\prod_{e\in E_5{}^1(G)} [e_G(u)+e_{L(G)}(e)]^2\,[e_G(u)+e_{L(G)}(e)]^2$$
$$\times \prod_{e\in E_5{}^2(G)} [e_G(u)+e_{L(G)}(e)]^2\,[e_G(u)+e_{L(G)}(e)]^2\times\ldots$$
$$\times \prod_{e\in E_5{}^7(G)} [e_G(u)+e_{L(G)}(e)]^2\begin{bmatrix}e_G(u)\\+e_{L(G)}(e)\end{bmatrix}^2$$
$$= [(5+5)^6\times(5+5)^6]\times[(5+7)^4\times(6+7)^4] \times[(5+6)^4\times(6+6)^4]\times[(7+7)^6\times(6+7)^6] \times[(8+6)^8\times(7+8)^8]\times[(7+8)^8\times(8+8)^8]\times [(7+9)^4\times(8+9)^4] = 5.0353\times e^{58}$$

(d) $$HB\prod_2 E(H_4(\beta,\beta)) = \prod_{ue}[e_G(u)e_{L(G)}(e)]^2$$
$$HB\prod_2 E(G) = \prod_{ue}[e_{H_4(\beta,\beta)}(u)e_{L(H_4(\beta,\beta))}(e)]^2$$
$$= \prod_{e=uv\in E_5(G)} [e_G(u)e_{L(G)}(e)]^2[e_G(v)e_{L(G)}(e)]^2\times$$
$$\prod_{e\in E_5{}^1(G)} [e_G(u)e_{L(G)}(e)]^2\,[e_G(u)e_{L(G)}(e)]^2$$
$$\times \prod_{e\in E_5{}^2(G)} [e_G(u)e_{L(G)}(e)]^2\,[e_G(u)+e_{L(G)}(e)]^2\times\ldots$$
$$\times \prod_{e\in E_5{}^7(G)} [e_G(u)e_{L(G)}(e)]^2\,[e_G(u)e_{L(G)}(e)]^2$$
$$= [(5\times5)^6\times(5\times5)^6]\times[(5\times7)^4(6\times7)^4] \times[(5\times6)^4\times(6\times6)^4]\times[(7\times7)^6\times(6\times7)^6] \times[(8\times6)^8\times(7\times8)^8]\times[(7\times8)^8\times(8\times8)^8]\times [(7\times9)^4\times(8\times9)^4] = 3.0070\times e^{69}$$

Similarly, let V_6 be the vertex set and E_6 be the edge set in $H_4(\beta,\gamma)$, then $|V_6| = 18$ and $|E_6| = 21$. Also the number of edges with eccentricities of end vertices are given as follows:

Let $E_6^{1} = \{e \in E(G) \ / \ e = uv \ , u,v \in V_6(G), e_G(u) = 5, e_G(v) = 5\}$

For $e \in E_6^{1}, e_{L(G)}(e) = 5$ and $|E_6^{1}| = 3$

$E_6^{2} = \{e \in E(G) \ / \ e = uv \ , u,v \in V_6(G), e_G(u) = 5, e_G(v) = 6\}$

For $e \in E_6^{2}, e_{L(G)}(e) = 6$ and $|E_6^{2}| = 4$ ...

$E_6^{6} = \{e \in E(G) \ / \ e = uv \ , u,v \in V_6(G), e_G(u) = 8, e_G(v) = 9\}$

For $e \in E_6^{6}, e_{L(G)}(e) = 9$ and $|E_6^{6}| = 2$

Hence

$$\text{(ii)(a) } B\prod_1 E(H_4(\beta,\gamma)) = \prod_{ue}[e_G(u) + e_{L(G)}(e)]$$

$$B\prod_1 E(G) = \prod_{ue}[e_{H_4(\beta,\gamma)}(u) + e_{L(H_4(\beta,\gamma))}(e)]$$

$$= \prod_{e=uv\in E_6(G)} [e_G(u) + e_{L(G)}(e)][e_G(v) + e_{L(G)}(e)]$$

$$= \prod_{e\in E_6^{1}(G)}[e_G(u) + e_{L(G)}(e)]\,[e_G(u) + e_{L(G)}(e)] \times \prod_{e\in E_6^{2}(G)}[e_G(u) + e_{L(G)}(e)]\,[e_G(u) + e_{L(G)}(e)] \times \ldots \times \prod_{e\in E_6^{6}(G)}[e_G(u) + e_{L(G)}(e)]\,[e_G(u) + e_{L(G)}(e)]$$

$$= [(5+5)^3(5+5)^3] \times [(5+6)^4(6+6)^4] \times [(8+8)^4(7+8)^4] \times [(9+8)^2(8+8)^2] \times [(7+7)^6(6+7)^6] \times [(8+9)^2(9+9)^2]$$

$$= 6.4313 \times e^{94}$$

$$\text{(b) } B\prod_2 E(H_4(\beta,\gamma)) = \prod_{ue}[e_G(u)e_{L(G)}(e)]$$

$$B\prod_1 E(G) = \prod_{ue}[e_{H_4(\beta,\gamma)}(u)e_{L(H_4(\beta,\gamma))}(e)]$$

$$= \prod_{e=uv\in E_6(G)}[e_G(u)e_{L(G)}(e)][e_G(v)e_{L(G)}(e)] = \prod_{e\in E_6^{1}(G)} [e_G(u)e_{L(G)}(e)]\,[e_G(u)e_{L(G)}(e)]$$

$$\times \prod_{e\in E_6^{2}(G)} [e_G(u)e_{L(G)}(e)]\,[e_G(u) + e_{L(G)}(e)] \times \ldots \times \prod_{e\in E_6^{6}(G)} [e_G(u)e_{L(G)}(e)]\,[e_G(u) + e_{L(G)}(e)]$$

$$= [(5\times5)^3(5\times5)^3] \times [(5\times6)^4(6\times6)^4] \times [(8\times8)^4(7\times8)^4] \times [(9\times8)^2(8\times8)^2] \times [(7\times7)^6(6\times7)^6] \times [(8\times9)^2(9\times9)^2]$$

$$= 4.4307 \times e^{140}$$

$$\text{(c) } HB\prod_1 E(H_4(\beta,\gamma)) = \prod_{ue}[e_G(u) + e_{L(G)}(e)]^2$$

$$HB\prod_1 E(G) = \prod_{ue}[e_{H_4(\beta,\gamma)}(u) + e_{L(H_4(\beta,\gamma))}(e)]^2$$

$$= \prod_{e=uv\in E_6(G)} [e_G(u) + e_{L(G)}(e)]^2[e_G(v) + e_{L(G)}(e)]^2 = \prod_{e\in E_6^{1}(G)} [e_G(u) + e_{L(G)}(e)]^2\,[e_G(u) + e_{L(G)}(e)]^2$$

$$\times \prod_{e\in E_6^{2}(G)} [e_G(u) + e_{L(G)}(e)]^2\,[e_G(u) + e_{L(G)}(e)]^2 \times \ldots \times \prod_{e\in E_6^{6}(G)} [e_G(u) + e_{L(G)}(e)]^2\,[e_G(u) + e_{L(G)}(e)]^2$$

$$= [(5+5)^6(5+5)^6] \times [(5+6)^8(6+6)^8] \times [(7+7)^{12}(6+7)^{12}] \times [(8+9)^4(9+9)^4] \times [(8+8)^8(7+8)^8] \times [(9+8)^4(8+8)^4]$$

$$= 2.5354 \times e^{63}$$

$$\text{(d) } HB\prod_2 E(H_4(\beta,\gamma)) = \prod_{ue}[e_G(u)e_{L(G)}(e)]^2$$

$$HB\prod_2 E(G) = \prod_{ue}[e_{H_4(\beta,\gamma)}(u)e_{L(H_4(\beta,\gamma))}(e)]^2$$

$$= \prod_{e=uv\in E_6(G)}[e_G(u)e_{L(G)}(e)]^2[e_G(v)e_{L(G)}(e)]^2 = \prod_{e\in E_6^{1}(G)} [e_G(u)e_{L(G)}(e)]^2\,[e_G(u)e_{L(G)}(e)]^2$$

$$\times \prod_{e\in E_6^{2}(G)} [e_G(u)e_{L(G)}(e)]^2\,[e_G(u) + e_{L(G)}(e)]^2 \times \ldots \times \prod_{e\in E_6^{6}(G)} [e_G(u)e_{L(G)}(e)]^2\,[e_G(u)e_{L(G)}(e)]^2$$

$$= [(5\times5)^6(5\times5)^6] \times [(5\times6)^8(6\times6)^8] \times [(7\times7)^{12}(6\times7)^{12}] \times [(8\times9)^4(9\times9)^4] \times [(8\times8)^8(7\times8)^8] \times [(9\times8)^4(8\times8)^4] = 2.9386 \times e^{95}$$

(iii) Consider the Zig-zag chain Z_4 in a line with $\propto$ –type, β – type and γ – type fusing and which is denoted as $Z_4(\gamma,\alpha)$, $Z_4(\gamma,\beta)$ and $Z_4(\gamma,\gamma)$. Fig 4 shows Z_4 with different types of fusing.

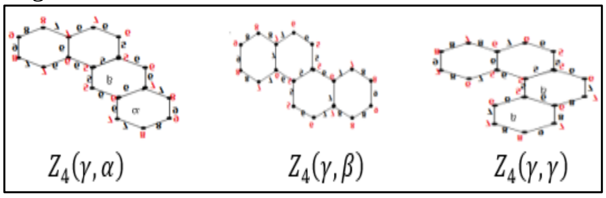


Fig. 4: Zig-zag chain Z_4, with α,β,γ type fusing

Let V_7 be the vertex set and E_7 be the edge set in $Z_4(\gamma,\alpha)$, then $|V_7| = 18$ and $|E_7| = 21$. Also $Z_4(\gamma,\alpha) = H_4(\beta,\alpha)$ since they are isomorphic .Hence we have

$$\text{(iii a)} B\prod_1 E(Z_4(\gamma,\alpha)) = B\prod_2 E(H_4(\beta,\alpha)) = 3.0449 \times e^{25}$$

$$\text{(b)} B\prod_2 E(Z_4(\gamma,\alpha)) = B\prod_1 E(H_4(\beta,\alpha)) = 2.6980 \times e^{90}$$

$$\text{(c)} HB\prod_1 E(Z_4(\gamma,\alpha)) = HB\prod_1 E(H_4(\beta,\alpha)) = 9.2334 \times e^{50}$$

$$\text{(d)} HB\prod_2 E(Z_4(\gamma,\alpha)) = HB\prod_2 E(H_4(\beta,\alpha)) = 9.3774 \times e^{72}$$

Let V_8 be the vertex set and E_8 be the edge set in $Z_4(\gamma,\beta)$, then $|V_8| = 18$ and $|E_8| = 21$. Here $Z_4(\gamma,\beta) = H_4(\beta,\beta)$.

Since they are isomorphic we have

$$\text{(iii a)} B\prod_1 E(Z_4(\gamma,\beta)) = B\prod_1 E(H_4(\beta,\beta)) = 2.6980 \times e^{90}$$

$$\text{(b)} B\prod_2 E(Z_4(\gamma,\beta)) = B\prod_2 E(H_4(\beta,\beta)) = 9.0362 \times e^{131}$$

$$\text{(c)} HB\prod_1 E(Z_4(\gamma,\beta)) = HB\prod_1 E(H_4(\beta,\beta)) = 5.0353 \times e^{58}$$

$$\text{(d)} HB\prod_2 E(Z_4(\gamma,\beta)) = HB\prod_2 E(H_4(\beta,\beta)) = 3.0070 \times e^{69}$$

Let V_9 be the vertex set and E_9 be the edge set in $Z_4(\gamma,\gamma)$, then $|V_9| = 18$ and $|E_9| = 21$. Here $Z_4(\gamma,\gamma) = H_4(\beta,\gamma)$. Since they are isomorphic we have

$$\text{(iii a)} B\prod_1 E(Z_4(\gamma,\gamma)) = B\prod_1 E(H_4(\beta,\gamma)) = 6.4313 \times e^{94}$$

$$\text{(b)} B\prod_2 E(Z_4(\gamma,\gamma)) = B\prod_2 E(H_4(\beta,\gamma)) = 4.4307 \times e^{140}$$

$$\text{(c)} HB\prod_1 E(Z_4(\gamma,\gamma)) = HB\prod_1 E(H_4(\beta,\gamma)) = 2.5354 \times e^{63}$$

$$\text{(d)} HB\prod_2 E(Z_4(\gamma,\gamma)) = HB\prod_2 E(H_4(\beta,\gamma)) = 2.9386 \times e^{95}$$

B. *Theorem 3.2*

Let L_5, H_5, and Z_5, aare the Linear chain with $\propto$ –type fusing, hexagonal chain with β – type fusing, and Zig-zag chain with γ – type fusing respectively, then

$$\text{(i)(a)} B\prod_1 E(L_5) = \begin{cases} 1.2600 \times e^{182}, \propto \text{–type fusing} \\ 1.3688 \times e^{129}, \beta \text{ – type fusing} \\ 1.3688 \times e^{129}, \gamma \text{ – type fusing} \end{cases}$$

$$\text{(b)} B\prod_2 E(L_5) = \begin{cases} 3.6466 \times e^{69}, \propto \text{–type fusing} \\ 1.0393 \times e^{188}, \beta \text{ – type fusing} \\ 1.0393 \times e^{188}, \gamma \text{ – type fusing} \end{cases}$$

$$(c) HB\prod_1 E(L_5) = \begin{cases} 1.0195 \times e^{94}, \propto -\text{type fusing} \\ 1.5489 \times e^{63}, \beta - \text{type fusing} \\ 1.5489 \times e^{63}, \gamma - \text{type fusing} \end{cases}$$

$$(d) HB\prod_2 E(L_5) = \begin{cases} 1.3298 \times e^{139}, \propto -\text{type fusing} \\ 1.0790 \times e^{95}, \beta - \text{type fusing} \\ 1.0790 \times e^{95}, \gamma - \text{type fusing} \end{cases}$$

$$(ii)(a) B\prod_1 E(H_5) = B\prod_1 E(Z_5)$$

$$= \begin{cases} 2.3992 \times e^{126}, \propto -\text{type fusing} \\ 4.7681 \times e^{112}, \beta - \text{type fusing} \\ 1.0972 \times e^{126}, \gamma - \text{type fusing} \end{cases}$$

$$(b) B\prod_2 E(H_5) = B\prod_2 E(Z_5)$$

$$= \begin{cases} 1.1643 \times e^{190}, \propto -\text{type fusing} \\ 1.2302 \times e^{184}, \beta - \text{type fusing} \\ 3.9157 \times e^{194}, \gamma - \text{type fusing} \end{cases}$$

$$(c) HB\prod_1 E(H_5) = HB\prod_1 E(Z_5)$$

$$= \begin{cases} 2.1836 \times e^{56}, \propto -\text{type fusing} \\ 1.0475 \times e^{63}, \beta - \text{type fusing} \\ 2.7697 \times e^{82}, \gamma - \text{type fusing} \end{cases}$$

$$(d) HB\prod_2 E(H_5) = HB\prod_2 E(Z_5)$$

$$= \begin{cases} 1.1092 \times e^{92}, \propto -\text{type fusing} \\ 1.9790 \times e^{97}, \beta - \text{type fusing} \\ 1.5622 \times e^{119}, \gamma - \text{type fusing} \end{cases}$$

1) Proof

Consider the Linear chain L_5, in a line with $\propto -$type, $\beta -$type and $\gamma -$type fusing and which is denoted as $L_5(\alpha,\alpha)$, $L_5(\alpha,\beta)$ and $L_5(\alpha,\gamma)$

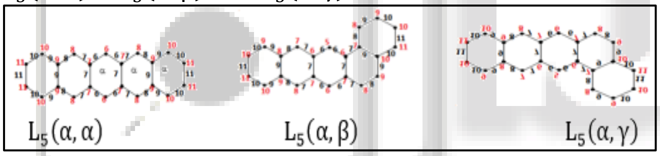


Fig. 5: Linear chain L_5 with α,β,γ type fusing

Let V_1 be the vertex set and E_1 be the edge set in $L_5(\alpha,\alpha)$, then $|V_1| = 22$ and $|E_1| = 26$. Also the number of edges with eccentricities of end vertices of G and L(G) are given as follows:

Let $E_1^{\,1} = \{e \in E(G)\ /\ e = uv\ , u, v \in V_1(G), e_G(u) = 11, e_G(v) = 11\}$

For $e \in E_1^{\,1}, e_{L(G)}(e) = 11$ and $|E_1^{\,1}| = 2$

$E_1^{\,2} = \{e \in E(G)\ /\ e = uv\ , u, v \in V_1(G), e_G(u) = 11, e_G(v) = 10\}$

For $e \in E_1^{\,2}, e_{L(G)}(e) = 10$ and $|E_1^{\,2}| = 4$ ...

$E_1^{\,8} = \{e \in E(G)\ /\ e = uv\ , u, v \in V_1(G), e_G(u) = 7, e_G(v) = 6\}$

For $e \in E_1^{\,8}, e_{L(G)}(e) = 7$ and $|E_1^{\,8}| = 4$. Hence

$$(i)(a) B\prod_1 E(L_5(\alpha,\alpha)) = \prod_{ue}[e_G(u) + e_{L(G)}(e)]$$

$$B\prod_1 E(G) = \prod_{ue}[e_{L_5(\alpha,\alpha)}(u) + e_{L(L_5(\alpha,\alpha))}(e)]$$

$$= \prod_{e=uv\in E_1(G)} [e_G(u) + e_{L(G)}(e)][e_G(v) + e_{L(G)}(e)]$$

$$= [(11+11)^2(11+11)^2] \times [(11+10)^4(10+10)^4] \times [(9+9)^2(9+9)^2] \times [(8+8)^4(8+9)^4] \times [(7+7)^4(8+7)^4] \times [(7+7)^2(7+7)^2] \times [(9+9)^4(10+9)^4] \times [(7+6)^4(6+6)^4] = 1.2600 \times e^{182}$$

$$(b) B\prod_2 E(L_5(\alpha,\alpha)) = \prod_{ue}[e_G(u) e_{L(G)}(e)]$$

$$B\prod_1 E(G) = \prod_{ue}[e_{L_5(\alpha,\alpha)}(u) e_{L(L_5(\alpha,\alpha))}(e)]$$

$$= \prod_{e=uv\in E_1(G)} [e_G(u) e_{L(G)}(e)][e_G(v) e_{L(G)}(e)]$$

$$= [(11\times 11)^2(11\times 11)^2] \times [(11\times 10)^4(10\times 10)^4] \times [(9\times 9)^2(9\times 9)^2] \times [(8\times 8)^4(8\times 9)^4] \times [(7\times 7)^4(8\times 7)^4] \times [(7\times 7)^2(7\times 7)^2] \times [(9\times 9)^4(10\times 9)^4] \times [(7\times 6)^4(6\times 6)^4] = 3.6466 \times e^{69}$$

$$(c)\ HB\prod_1 E(L_5(\alpha,\alpha)) = \prod_{ue}[e_G(u) + e_{L(G)}(e)]^2$$

$$HB\prod_1 E(G) = \prod_{ue}[e_{L_5(\alpha,\alpha)}(u) + e_{L(L_5(\alpha,\alpha))}(e)]^2$$

$$= \prod_{e=uv\in E_1(G)} [e_G(u) + e_{L(G)}(e)]^2[e_G(v) + e_{L(G)}(e)]^2$$

$$= [(11+11)^4(11+11)^4] \times [(11+10)^8(10+10)^8] \times [(9+9)^4(9+9)^4] \times [(8+8)^8(8+9)^8] \times [(7+7)^8(8+7)^8] \times [(7+7)^4(7+7)^4] \times [(9+9)^8(10+9)^8] \times [(7+6)^8(6+6)^8] = 1.0195 \times e^{94}$$

$$(d)\ HB\prod_2 E(L_5(\alpha,\alpha)) = \prod_{ue}[e_G(u) e_{L(G)}(e)]^2$$

$$HB\prod_2 E(G) = \prod_{ue}[e_{L_5(\alpha,\alpha)}(u) e_{L(L_5(\alpha,\alpha))}(e)]^2$$

$$= \prod_{e=uv\in E_1(G)} [e_G(u) e_{L(G)}(e)]^2[e_G(v) e_{L(G)}(e)]^2$$

$$= [(11\times 11)^4(11\times 11)^4] \times [(11\times 10)^8(10\times 10)^8] \times [(9\times 9)^4(9\times 9)^4] \times [(8\times 8)^8(8\times 9)^8] \times [(7\times 7)^8(8\times 7)^8] \times [(7\times 7)^4(7\times 7)^4] \times [(9\times 9)^8(10\times 9)^8] \times [(7\times 6)^8(6\times 6)^8] = 1.3298 \times e^{139}$$

Also let V_2 be the vertex set and E_2 be the edge set in $L_5(\alpha,\beta)$, then $|V_2| = 22$ and $|E_2| = 26$.

The number of edges with eccentricities of end vertices of G and L(G) are given as follows:

Let

$E_2^{\,1} = \{e \in E(G)\ /\ e = uv\ , u, v \in V_2(G), e_G(u) = 5, e_G(v) = 6\}$

For $e \in E_2^{\,1}, e_{L(G)}(e) = 6$ and $|E_2^{\,1}| = 2$

$E_2^{\,2} = \{e \in E(G)\ /\ e = uv\ , u, v \in V_2(G), e_G(u) = 7, e_G(v) = 6\}$

For $e \in E_2^{\,2}, e_{L(G)}(e) = 7$ and $|E_2^{\,2}| = 4$ ...

$E_2^{\,12} = \{e \in E(G)\ /\ e = uv\ , u, v \in V_2(G), e_G(u) = 10, e_G(v) = 11\}$

For $e \in E_2^{\,12}, e_{L(G)}(e) = 10$ and $|E_2^{\,12}| = 2$. Hence

$$(i)(a)\ B\prod_1 E(L_5(\alpha,\beta)) = \prod_{ue}[e_G(u) + e_{L(G)}(e)]$$

$$B\prod_1 E(G) = \prod_{ue}[e_{L_5(\alpha,\beta)}(u) + e_{L(L_5(\alpha,\beta))}(e)]$$

$$= \prod_{e=uv\in E_2(G)} [e_G(u) + e_{L(G)}(e)][e_G(v) + e_{L(G)}(e)]$$

$$= [(5+6)^2(6+6)^2] \times [(6+6)^2(7+6)^2] \times [(7+6)^4(7+7)^4] \times [(7+9)(10+9)] \times [(7+7)^2(8+7)^2] \times [(7+8)^2(8+8)^2] \times [(9+8)^2(8+8)^2] \times [(9+8)^3(9+9)^3] \times [(9+10)^2(10+10)^2] \times [(11+10)^2(11+11)^2] \times [(9+10)^2(9+9)^2] \times [(11+10)^2(10+10)^2] = 1.3688 \times e^{129}$$

$$(b)\ B\prod_2 E(L_5(\alpha,\beta)) = \prod_{ue}[e_G(u) e_{L(G)}(e)]$$

$$B\prod_2 E(G) = \prod_{ue}[e_{L_5(\alpha,\beta)}(u) e_{L(L_5(\alpha,\beta))}(e)]$$

$$= \prod_{e=uv\in E_2(G)} [e_G(u) + e_{L(G)}(e)][e_G(v) e_{L(G)}(e)]$$

$$= [(5\times 6)^2(6\times 6)^2] \times [(6\times 6)^2(7\times 6)^2] \times [(7\times 6)^4(7\times 7)^4] \times [(7\times 9)(10\times 9)] \times [(7\times 7)^2(8\times 7)^2] \times [(7\times 8)^2(8\times 8)^2]$$

$$\times [(9\times 8)^2(8\times 8)^2]\times[(9\times 8)^3(9\times 9)^3]$$
$$\times [(9\times 10)^2(10\times 10)^2]\times[(11\times 10)^2(11\times 11)^2]$$
$$\times [(9\times 10)^2(9\times 9)^2]\times[(11\times 10)^2(10\times 10)^2]$$
$$= 1.0393\times e^{188}$$

(c) $HB\prod_1 E(L_5(\alpha,\beta)) = \prod_{ue}[e_G(u)+e_{L(G)}(e)]^2$

$$HB\prod_1 E(G) = \prod_{ue}[e_{L_5(\alpha,\beta)}(u)+e_{L(L_5(\alpha,\beta))}(e)]^2$$
$$= \prod_{e=uv\in E_2(G)}[e_G(u)+e_{L(G)}(e)]^2[e_G(v)+e_{L(G)}(e)]^2$$
$$= [(5+6)^4(6+6)^4]\times[(6+6)^4(7+6)^4]$$
$$\times[(7+6)^8(7+7)^8]\times[(7+9)^2(10+9)^2]$$
$$\times[(7+7)^4(8+7)^4]\times[(7+8)^4(8+8)^4]$$
$$\times[(9+8)^4(8+8)^4]\times[(9+8)^6(9+9)^6]$$
$$\times[(9+10)^4(10+10)^4]\times[(11+10)^4(11+11)^4]$$
$$\times[(9+10)^4(9+9)^4]\times[(11+10)^4(10+10)^4]$$
$$= 1.5489\times e^{63}$$

(d) $HB\prod_2 E(L_5(\alpha,\beta)) = \prod_{ue}[e_G(u)e_{L(G)}(e)]^2$

$$HB\prod_2 E(G) = \prod_{ue}[e_{L_5(\alpha,\beta)}(u)e_{L(L_5(\alpha,\beta))}(e)]^2$$
$$= \prod_{e=uv\in E_2(G)}[e_G(u)e_{L(G)}(e)]^2[e_G(v)e_{L(G)}(e)]^2 =$$
$$[(5\times 6)^4(6\times 6)^4]\times[(6\times 6)^4(7\times 6)^4]$$
$$\times[(7\times 6)^8(7\times 7)^8]\times[(11\times 10)^4(11\times 11)^4]$$
$$\times[(7\times 7)^4(8\times 7)^4]\times[(7\times 9)^2(10\times 9)^2]$$
$$\times[(9\times 10)^4(10\times 10)^4]\times[(9\times 8)^4(8\times 8)^4]$$
$$\times[(9\times 8)^6(9\times 9)^6]\times[(9\times 10)^4(9\times 9)^4]$$
$$\times[(11\times 10)^4(10\times 10)^4]\times[(7\times 8)^4(8\times 8)^4]$$
$$= 1.0790\times e^{95}$$

Let V_3 be the vertex set and E_3 be the edge set in $L_5(\alpha,\gamma)$, then $|V_3| = 22$ and $|E_3| = 26$. Also $L_5(\alpha,\gamma) = L_5(\alpha,\beta)$ since they are isomorphic .Hence we have

(i) (a) $B_1E(L_5(\alpha,\gamma)) = B_1E(L_5(\alpha,\beta)) = 1.3688\times e^{129}$

(b) $B_2E(L_5(\alpha,\gamma)) = B_2E(L_5(\alpha,\beta)) = 1.0393\times e^{188}$

(c) $HB_1E(L_5(\alpha,\gamma)) = HB_1E(L_5(\alpha,\beta)) = 1.5489\times e^{63}$

(d) $HB_2E(L_5(\alpha,\gamma)) = HB_2E(L_5(\alpha,\beta)) = 1.0790\times e^{95}$

(ii) Consider the Hexagonal chain H_5, in a line with $\propto$ −type, β − type and γ − type fusing and which is denoted as $H_5(\beta,\alpha)$, $H_5(\beta,\beta)$ and $H_5(\beta,\gamma)$.Fig 6. Shows H_5 with different types of fusing.

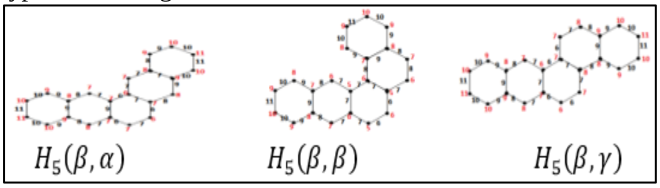


Fig. 6: Hexogonal chain H_5, with α,β,γ type fusing

Let V_4 be the vertex set and E_4 be the edge set in $H_5(\beta,\alpha)$, then $|V_4| = 22$ and $|E_4| = 26$. Also the number of edges with eccentricities of end vertices are given as follows:

Let

$E_4{}^1 = \{e\in E(G)\ /\ e = uv\ , u,v\in V_4(G), e_G(u) = 11, e_G(v) = 10\}$

For $e\in E_4{}^1, e_{L(G)}(e) = 11$ and $|E_4{}^1| = 2$

$E_4{}^2 = \{e\in E(G)\ /\ e = uv\ , u,v\in V_4(G), e_G(u) = 11, e_G(v) = 10\}$

For $e\in E_4{}^2, e_{L(G)}(e) = 10$ and $|E_4{}^2| = 2$ ...

$E_4{}^{11} = \{e\in E(G)\ /\ e = uv\ , u,v\in V_4(G), e_G(u) = 7, e_G(v) = 7\}$

For $e\in E_4{}^{11}, e_{L(G)}(e) = 7$ and $|E_4{}^{11}| = 1$.

Hence

(ii)(a) $B\prod_1 E(H_5(\beta,\alpha)) = \prod_{ue}[e_G(u)+e_{L(G)}(e)]$

$$B\prod_1 E(G) = \prod_{ue}[e_{H_5(\beta,\alpha)}(u)+e_{L(H_5(\beta,\alpha))}(e)]$$
$$= \prod_{e=uv\in E_4(G)}[e_G(u)+e_{L(G)}(e)][e_G(v)+e_{L(G)}(e)]$$
$$= [(7+6)^3(6+6)^3]\times[(7+6)^2(7+7)^2]$$
$$\times[(7+7)\times(7+7)]\times[(8+8)^2(9+8)^2]$$
$$\times[(7+7)^4(8+7)^4]\times[(8+8)\times(8+8)]$$
$$\times[(10+10)^2(9+10)^2]\times[(10+10)^2(11+10)^2]$$
$$\times[(8+9)^5(9+9)^5]\times[(10+9)^2\times(9+9)^2]$$
$$\times[(11+11)^2(11+10)^2] = 2.3992\times e^{126}$$

(b) $B\prod_2 E(H_5(\beta,\alpha)) = \prod_{ue}[e_G(u)e_{L(G)}(e)]$

$$B\prod_2 E(G) = \prod_{ue}[e_{H_5(\beta,\alpha)}(u)e_{L(H_5(\beta,\alpha))}(e)]$$
$$= \prod_{e=uv\in E_4(G)}[e_G(u)e_{L(G)}(e)][e_G(v)e_{L(G)}(e)]$$
$$= [(7\times 6)^3(6\times 6)^3]\times[(7\times 6)^2(7\times 7)^2]$$
$$\times[(7\times 7)^4(8\times 7)^4]\times[(8\times 8)\times(8\times 8)]$$
$$\times[(7\times 7)\times(7\times 7)]\times[(8\times 8)^2(9\times 8)^2]$$
$$\times[(10\times 10)^2(9\times 10)^2]\times[(10\times 10)^2(11\times 10)^2]$$
$$\times[(8\times 9)^5(9\times 9)^5]\times[(10\times 9)^2\times(9\times 9)^2]$$
$$\times[(11\times 11)^2(11\times 10)^2] = 1.1643\times e^{190}$$

(c) $HB\prod_1 E(H_5(\beta,\alpha)) = \prod_{ue}[e_G(u)+e_{L(G)}(e)]^2$

$$HB\prod_1 E(G) = \prod_{ue}[e_{H_5(\beta,\alpha)}(u)+e_{L(H_5(\beta,\alpha))}(e)]^2$$
$$= \prod_{e=uv\in E_4(G)}[e_G(u)+e_{L(G)}(e)]^2[e_G(v)+e_{L(G)}(e)]^2$$
$$= [(7+6)^6(6+6)^6]\times[(7+6)^4(7+7)^4]$$
$$\times[(7+7)^8(8+7)^8]\times[(8+8)^1\times(8+8)^2]$$
$$\times[(7+7)^2\times(7+7)^2]\times[(8+8)^4(9+8)^4]$$
$$\times[(10+10)^4(9+10)^4]\times[(10+10)^4(11+10)^4]$$
$$\times[(8+9)^{10}(9+9)^{10}]\times[(10+9)^4\times(9+9)^4]$$
$$\times[(11+11)^4(11+10)^4] = 2.1836\times e^{56}$$

(d) $HB\prod_2 E(H_5(\beta,\alpha)) = \prod_{ue}[e_G(u)+e_{L(G)}(e)]^2$

$$HB\prod_2 E(G) = \prod_{ue}[e_{H_5(\beta,\alpha)}(u)+e_{L(H_5(\beta,\alpha))}(e)]^2$$
$$= \prod_{e=uv\in E_4(G)}[e_G(u)e_{L(G)}(e)]^2[e_G(v)e_{L(G)}(e)]^2$$
$$= [(7\times 6)^6(6\times 6)^6]\times[(7\times 6)^4(7\times 7)^4]$$
$$\times[(7\times 7)^8(8\times 7)^8]\times[(8\times 8)^1\times(8\times 8)^2]$$
$$\times[(7\times 7)^2\times(7\times 7)^2]\times[(8\times 8)^4(9\times 8)^4]$$
$$\times[(10\times 10)^4(9\times 10)^4]\times[(10\times 10)^4(11\times 10)^4]$$
$$\times[(8\times 9)^{10}(9\times 9)^{10}]\times[(10\times 9)^4\times(9\times 9)^4]$$
$$\times[(11\times 11)^4(11\times 10)^4] = 1.1092\times e^{92}$$

Also let V_5 be the vertex set and E_5 be the edge set in $H_5(\beta,\beta)$, then $|V_5| = 22$ and $|E_5| = 26$. The number of edges with eccentricities of end vertices are given as follows:

Let $E_5{}^1 = \{e\in E(G)\ /\ e = uv\ , u,v\in V_5(G),\ e_G(u) = 5, e_G(v) = 6\}$

For $e\in E_5{}^1, e_{L(G)}(e) = 7$ and $|E_5{}^1| = 4$

$E_5{}^2 = \{e\in E(G)\ /\ e = uv\ , u,v\in V_5(G), e_G(u) = 5, e_G(v) = 6\}$

For $e\in E_5{}^2, e_{L(G)}(e) = 6$ and $|E_5{}^2| = 2$ ...

$E_5{}^{10} = \{e\in E(G)\ /\ e = uv\ , u,v\in V_5(G), e_G(u) = 9, e_G(v) = 8\}$

For $e\in E_5{}^{10}, e_{L(G)}(e) = 11$ and $|E_5{}^{10}| = 2$. Hence

(ii)(a) $B\prod_1 E(H_5(\beta,\beta)) = \prod_{ue}[e_G(u)+e_{L(G)}(e)]$

$$B\prod_1 E(G) = \prod_{ue}[e_{H_5(\beta,\beta)}(u)+e_{L(H_5(\beta,\beta))}(e)]$$
$$= \prod\nolimits_{e=uv\in E_5(G)}[e_G(u)+e_{L(G)}(e)][e_G(v)+e_{L(G)}(e)] =$$
$$[(5+7)^4\times(6+7)^4]\times[(5+6)^2\times(6+6)^2]$$

$$\times [(7+8)^2(8+8)^2] \times [(7+7)^2(6+7)^2]$$
$$\times [(7+8)^4(6+8)^4] \times [(7+9)^4(8+9)^4]$$
$$\times [(8+9)^2(9+9)^2] \times [(8+10)^2+(9+10)^2]$$
$$\times [(9+10)^2(10+10)^2] \times [(11+9)^2(11+10)^2]$$
$$= 4.7681 \times e^{112}$$

$$\text{(b) } B\prod_2 E(H_5(\beta,\beta)) = \prod_{ue}[e_G(u)e_{L(G)}(e)]$$
$$B\prod_2 E(G) = \prod_{ue}[e_{H_5(\beta,\beta)}(u)e_{L(H_5(\beta,\beta))}(e)]$$
$$= \prod_{e=uv\in E_5(G)} [e_G(u)e_{L(G)}(e)][e_G(v)e_{L(G)}(e)] =$$
$$[(5\times 7)^4 \times (6\times 7)^4] \times [(5\times 6)^2 \times (6\times 6)^2]$$
$$\times [(7\times 8)^2(8\times 8)^2] \times [(7\times 7)^2(6\times 7)^2]$$
$$\times [(7\times 8)^4(6\times 8)^4] \times [(7\times 9)^4(8\times 9)^4]$$
$$\times [(9\times 10)^2(10\times 10)^2] \times [(8\times 9)^2(9\times 9)^2]$$
$$\times [(8\times 10)^2(9\times 10)^2] \times [(11\times 9)^2(11\times 10)^2]$$
$$= 1.2302 \times e^{184}$$

$$\text{(c) } HB\prod_1 E(H_5(\beta,\beta)) = \prod_{ue}[e_G(u)+e_{L(G)}(e)]^2$$
$$HB\prod_1 E(G) = \prod_{ue}[e_{H_5(\beta,\beta)}(u)+e_{L(H_5(\beta,\beta))}(e)]^2$$
$$= \prod_{e=uv\in E_5(G)}[e_G(u)+e_{L(G)}(e)]^2[e_G(v)+e_{L(G)}(e)]^2 =$$
$$[(5+7)^8 \times (6+7)^8] \times [(5+6)^4 \times (6+6)^4]$$
$$\times [(7+8)^4(8+8)^4] \times [(7+7)^4(6+7)^4]$$
$$\times [(7+8)^8(6+8)^8] \times [(7+9)^8(8+9)^8]$$
$$\times [(8+9)^4(9+9)^4] \times [(8+10)^4(9+10)^4]$$
$$\times [(9+10)^4(10+10)^4] \times [(11+9)^4(11+10)^4]$$
$$= 1.0475 \times e^{63}$$

$$\text{(d) } HB\prod_2 E(H_5(\beta,\beta)) = \prod_{ue}[e_G(u)e_{L(G)}(e)]^2$$
$$HB\prod_2 E(G) = \prod_{ue}[e_{H_5(\beta,\beta)}(u)e_{L(H_5(\beta,\beta))}(e)]^2$$
$$= \prod_{e=uv\in E_5(G)} [e_G(u)e_{L(G)}(e)]^2[e_G(v)e_{L(G)}(e)]^2 =$$
$$[(5\times 7)^8 \times (6\times 7)^8] \times [(5\times 6)^4 \times (6\times 6)^4]$$
$$\times [(7\times 8)^4(8\times 8)^4] \times [(7\times 7)^4(6\times 7)^4]$$
$$\times [(7\times 8)^8(6\times 8)^8] \times [(7\times 9)^8(8\times 9)^8]$$
$$\times [(8\times 9)^4(9\times 9)^4] \times [(8\times 10)^4(9\times 10)^4]$$
$$\times [(9\times 10)^4(10\times 10)^4] \times [(11\times 9)^4(11\times 10)^4]$$
$$= 1.9790 \times e^{97}$$

Similarly, let V_6 be the vertex set and E_6 be the edge set in $H_5(\beta,\gamma)$, then $|V_6| = 22$ and $|E_6| = 26$. Also the number of edges with eccentricities of end vertices are given as follows: Let

$E_6{}^1 = \{e \in E(G) \ / \ e = uv\ , u, v \in V_6(G),\quad e_G(u) = 10, e_G(v) = 11\}$

For $e \in E_6{}^1, e_{L(G)}(e) = 10$ and $|E_6{}^1| = 2$

$E_6{}^2 = \{e \in E(G) \ / \ e = uv\ , u, v \in V_6(G),\quad e_G(u) = 10, e_G(v) = 11\}$

For $e \in E_6{}^2, e_{L(G)}(e) = 11$ and $|E_6{}^2| = 2$

...

$E_6{}^{12} = \{e \in E(G) \ / \ e = uv\ , u, v \in V_6(G), e_G(u) = 7, e_G(v) = 7\}$

For $e \in E_6{}^{12}, e_{L(G)}(e) = 6$ and $|E_6{}^{12}| = 1$. Hence

$$\text{(ii)(a) } B\prod_1 E(H_5(\beta,\gamma)) = \prod_{ue}[e_G(u)+e_{L(G)}(e)]$$
$$B\prod_1 E(G) = \prod_{ue}[e_{H_5(\beta,\gamma)}(u)+e_{L(H_5(\beta,\gamma))}(e)] =$$
$$\prod_{e=uv\in E_6(G)}[e_G(u)+e_{L(G)}(e)][e_G(v)+e_{L(G)}(e)]$$
$$= [(10+11)^2(10+10)^2] \times [(10+11)^2(11+11)^2]$$
$$\times [(10+9)^2(10+10)^2] \times [(9+9)^2(10+9)^2]$$
$$\times [(8+9)^2(8+8)^2] \times [(8+9)^4(9+9)^4]$$
$$\times [(8+7)(8+8)] \times [(8+7)^4(7+7)^4]$$
$$\times [(8+8)(8+8)] \times [(6+7)^2(7+7)^2]$$
$$\times [(6+7)^3(6+6)^3] \times [(6+7)(6+7)] = 1.0972 \times e^{126}$$

$$\text{(b) } B\prod_2 E(H_5(\beta,\gamma)) = \prod_{ue}[e_G(u)e_{L(G)}(e)]$$
$$B\prod_2 E(G) = \prod_{ue}[e_{H_5(\beta,\gamma)}(u)e_{L(H_5(\beta,\gamma))}(e)]$$
$$= \prod_{e=uv\in E_6(G)} [e_G(u)e_{L(G)}(e)][e_G(v)e_{L(G)}(e)] =$$
$$[(10\times 11)^2(10\times 10)^2] \times [(10\times 11)^2 \times^2]$$
$$\times [(10\times 9)^2(10\times 10)^2] \times [(9\times 9)^2(10\times 9)^2]$$
$$\times [(8\times 9)^2(8\times 8)^2] \times [(8\times 9)^4(9\times 9)^4]$$
$$\times [(8\times 7)(8\times 8)] \times [(8\times 7)^4(7\times 7)^4]$$
$$\times [(8\times 8)(8\times 8)] \times [(6\times 7)^2(7\times 7)^2]$$
$$\times [(6\times 7)^3(6\times 6)^3] \times [(6\times 7)(6\times 7)] = 3.9157 \times e^{194}$$

$$\text{(c) } HB\prod_1 E(H_5(\beta,\gamma)) = \prod_{ue}[e_G(u)+e_{L(G)}(e)]^2$$
$$HB\prod_1 E(G) = \prod_{ue}[e_{H_5(\beta,\gamma)}(u)+e_{L(H_5(\beta,\gamma))}(e)]^2$$
$$= \prod_{e=uv\in E_6(G)}[e_G(u)+e_{L(G)}(e)]^2[e_G(v)+e_{L(G)}(e)]^2 =$$
$$[(10+11)^4(10+10)^4] \times [(10+11)^4(11+11)^4]$$
$$\times [(10+9)^4(10+10)^4] \times [(9+9)^4(10+9)^4]$$
$$\times [(8+9)^4(8+8)^4] \times [(8+9)^8(9+9)^8]$$
$$\times [(8+7)^2(8+8)^2] \times [(8+7)^8(7+7)^8]$$
$$\times [(8+8)^2(8+8)^2] \times [(6+7)^4(7+7)^4]$$
$$\times [(6+7)^6(6+6)^6] \times [(6+7)^2(6+7)^2] = 2.7697 \times e^{82}$$

$$\text{(d) } HB\prod_2 E(H_5(\beta,\gamma)) = \prod_{ue}[e_G(u)+e_{L(G)}(e)]^2$$
$$HB\prod_2 E(G) = \prod_{ue}[e_{H_5(\beta,\gamma)}(u)+e_{L(H_5(\beta,\gamma))}(e)]^2$$
$$= \prod_{e=uv\in E_6(G)} [e_G(u)e_{L(G)}(e)]^2[e_G(v)e_{L(G)}(e)]^2$$
$$= [(10\times 11)^4(10\times 10)^4] \times [(10\times 11)^4(11\times 11)^4]$$
$$\times [(10\times 9)^4(10\times 10)^4] \times [(9\times 9)^4(10\times 9)^4]$$
$$\times [(8\times 9)^4(8\times 8)^4] \times [(8\times 9)^8(9\times 9)^8]$$
$$\times [(8\times 7)^2(8\times 8)^2] \times [(8\times 7)^8(7\times 7)^8]$$
$$\times [(8\times 8)^2(8\times 8)^2] \times [(6\times 7)^4(7\times 7)^4]$$
$$\times [(6\times 7)^6(6\times 6)^6] \times [(6\times 7)^2(6\times 7)^2] = 1.5622 \times e^{119}$$

Let V_7 be the vertex set and E_7 be the edge set in $Z_5(\gamma,\alpha)$, then $|V_7| = 22$ and $|E_7| = 2$. Also $Z_5(\gamma,\alpha) = H_5(\beta,\alpha)$. Since they are isomorphic, we have

(iii) (a) $B_1E(Z_5(\gamma,\alpha)) = B_1E(H_5(\beta,\alpha)) = 2.3992 \times e^{126}$

(b) $B_2E(Z_5(\gamma,\alpha)) = B_2E(H_5(\beta,\alpha)) = 1.1643 \times e^{190}$

(c) $HB_1E(Z_5(\gamma,\alpha)) = HB_1E(H_5(\beta,\alpha)) = 2.1836 \times e^{56}$

(d) $HB_2E(Z_5(\gamma,\alpha)) = HB_2E(H_5(\beta,\alpha)) = 1.1092 \times e^{92}$

Let V_8 be the vertex set and E_8 be the edge set in $Z_5(\gamma,\beta)$, then $|V_8| = 22$ and $|E_8| = 26$. Also $Z_5(\gamma,\beta) = H_5(\beta,\gamma)$. Since they are isomorphic, we have

$$\text{(iii)(a)} B\prod_1 E(Z_5(\gamma,\beta)) = B\prod_1 E(H_5(\beta,\gamma)) = 4.7681 \times e^{112}$$

$$\text{(b)} B\prod_2 E(Z_5(\gamma,\beta)) = B\prod_2 E(H_5(\beta,\gamma)) = 1.2302 \times e^{184}$$

$$\text{(c)} HB\prod_1 E(Z_5(\gamma,\beta)) = HB\prod_1 E(H_5(\beta,\gamma)) = 1.0475 \times e^{63}$$

$$\text{(d)} HB\prod_2 E(Z_5(\gamma,\beta)) = HB\prod_2 E(H_5(\beta,\gamma)) = 1.9790 \times e^{97}$$

Let V_9 be the vertex set and E_9 be the edge set in $Z_5(\gamma,\gamma)$, then $|V_9| = 22$ and $|E_9| = 26$. Also $Z_5(\gamma,\gamma) = H_5(\beta,\beta)$. Since they are isomorphic, we have

(iii)(a) $B_1E(Z_5(\gamma,\gamma)) = B_1E(H_5(\beta,\beta)) = 1.0972 \times e^{126}$

(b) $B_2E(Z_5(\gamma,\gamma)) = B_2E(H_5(\beta,\beta)) = 3.9157 \times e^{194}$

(c) $HB_1E(Z_5(\gamma,\gamma)) = HB_1E(H_5(\beta,\beta)) = 2.7697 \times e^{82}$

(d) $HB_2E(Z_5(\gamma,\gamma)) = H\,B_2E(H_5(\beta,\beta)) = 1.5622 \times e^{119}$

C. Theorem 2.3

If L_6 is a Linear chain with $\propto -$type fusing, H_6 is the Hexagonal chain with $\beta -$ type fusing, and Z_6 is a Zig-zag chain with $\gamma -$ type fusing then

$$\text{(i)(a)}B\prod_1 E(L_6) = 7.6715 \times e^{164}$$

$$\text{(b)}B\prod_2 E(L_6) = 2.4405 \times e^{238}$$

$$\text{(c)}HB\prod_1 E(L_6) = 2.7278 \times e^{876}$$

$$\text{(d)}HB\prod_2 E(L_6) = 2.4259 \times e^{126}$$

$$\text{(ii)(a)}B\prod_1 E(H_6) = 7.4408 \times e^{15}$$

$$\text{(b)}B\prod_2 E(H_6) = 5.8852 \times e^{252}$$

$$\text{(c)}HB\prod_1 E(H_6) = 3.2309 \times e^{77}$$

$$\text{(d)}HB\prod_2 E(H_6) = 1.1063 \times e^{123}$$

$$\text{(iii)(a)}B\prod_1 E(Z_6) = 1.0439 \times e^{155}$$

$$\text{(b)}B\prod_2 E(Z_6) = 1.2056 \times e^{198}$$

$$\text{(c)}HB\prod_1 E(Z_6) = 1.2568 \times e^{175}$$

$$\text{(d)}HB\prod_2 E(Z_6) = 1.2238 \times e^{246}$$

1) Proof

Consider the Linear chain L_6 as in the diagram

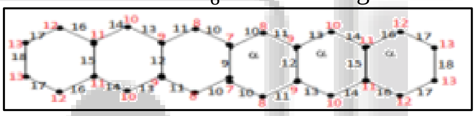


Fig. 7: Linear chain L_6

Let V_1 be the vertex set and E_1 be the edge set in L_6, then $|V_1| = 26$ and $|E_1| = 31$. Also the number of edges with eccentricities of end vertices of G and L(G) are given as follows: Let

$E_1{}^1 = \{e \in E(G)\ /\ e = uv\ , u, v \in V_1(G),\quad e_G(u) = 13, e_G(v) = 13\}$

For $e \in E_1{}^1, e_{L(G)}(e) = 18$ and $|E_1{}^1| = 2$

$E_1{}^2 = \{e \in E(G)\ /\ e = uv\ , u, v \in V_1(G),\quad e_G(u) = 12, e_G(v) = 13\}$

For $e \in E_1{}^2, e_{L(G)}(e) = 17$ and $|E_1{}^2| = 4$

...

$E_1{}^{10} = \{e \in E(G)\ /\ e = uv\ , u, v \in V_1(G), e_G(u) = 7, e_G(v) = 7\}$

For $e \in E_1{}^{10}, e_{L(G)}(e) = 9$ and $|E_1{}^{10}| = 1$. Hence

$$\text{(ii)(a) } B\prod_1 E(L_6) = \prod_{ue}[e_G(u) + e_{L(G)}(e)]$$

$$B\prod_1 E(G) = \prod_{ue}[e_{L_6}(u) + e_{L(L_6)}(e)]$$

$$= \prod_{e=uv\in E_1(G)} [e_G(u) + e_{L(G)}(e)][e_G(v) + e_{L(G)}(e)]$$

$$= [(13+18)^2(13+18)^2] \times [(12+17)^4(13+17)^4] \times [(11+16)^4(12+16)^4] \times [(9+13)^4(10+13)^4] \times [(9+12)^2 + (9+12)^2] \times [(11+8)^4(11+9)^4] \times [(10+14)^4(11+14)^4] \times [(11+15)^2(11+15)^2] \times [(10+7)^4(10+8)^4] \times [(9+7)(9+7)] = 7.6715 \times e^{164}$$

$$\text{(b) } B\prod_2 E(L_6) = \prod_{ue}[e_G(u)e_{L(G)}(e)]$$

$$B\prod_2 E(G) = \prod_{ue}[e_{L_6}(u)e_{L(L_6)}(e)]$$

$$= \prod_{e=uv\in E_1(G)} [e_G(u)e_{L(G)}(e)][e_G(v)e_{L(G)}(e)]$$

$$= [(13\times18)^2(13\times18)^2] \times [(12\times17)^4(13\times17)^4] \times [(11\times16)^4(12\times16)^4] \times [(9\times13)^4(10\times13)^4] \times [(9\times12)^2 + (9\times12)^2] \times [(11\times8)^4(11\times9)^4] \times [(10\times14)^4(11\times14)^4] \times [(11\times15)^2(11\times15)^2] \times [(10\times7)^4(10\times8)^4] \times [(9\times7)(9\times7)] = 2.4405 \times e^{238}$$

$$\text{(c) } HB\prod_1 E(L_6) = \prod_{ue}[e_G(u) + e_{L(G)}(e)]^2$$

$$HB\prod_1 E(G) = \prod_{ue}[e_{L_6}(u) + e_{L(L_6)}(e)]^2$$

$$= \prod_{e=uv\in E_1(G)} [e_G(u) + e_{L(G)}(e)]^2[e_G(v) + e_{L(G)}(e)]^2$$

$$= [(13+18)^4(13+18)^4] \times [(12+17)^8(13+17)^8] \times [(11+16)^8(12+16)^8] \times [(9+13)^8(10+13)^8] \times [(9+12)^4 + (9+12)^4] \times [(11+8)^8(11+9)^8] \times [(10+14)^8(11+14)^8] \times [(11+15)^4(11+15)^4] \times [(10+7)^8(10+8)^8] \times [(9+7)(9+7)] = 2.7278 \times e^{876}$$

$$\text{(d) } HB\prod_2 E(L_6) = \prod_{ue}[e_G(u)e_{L(G)}(e)]^2$$

$$HB\prod_2 E(G) = \prod_{ue}[e_{L_6}(u)e_{L(L_6)}(e)]^2$$

$$= \prod_{e=uv\in E_1(G)} [e_G(u)e_{L(G)}(e)]^2[e_G(v)e_{L(G)}(e)]^2$$

$$= [(13\times18)^4(13\times18)^4] \times [(12\times17)^8(13\times17)^8] \times [(11\times16)^8(12\times16)^8] \times [(9\times13)^8(10\times13)^8] \times [(9\times12)^4 + (9\times12)^4] \times [(11\times8)^8(11\times9)^8] \times [(10\times14)^8(11\times14)^8] \times [(11\times15)^4(11\times15)^4] \times [(10\times7)^8(10\times8)^8] \times [(9\times7)(9\times7)] = 2.4259 \times e^{126}$$

Consider the Hexagonal chain H_6 as in the diagram

Fig. 8: A Hexagonal chain H_6

Let V_2 be the vertex set and E_2 be the edge set in H_6, then $|V_2| = 26$ and $|E_2| = 31$

Also the number of edges with eccentricities of end vertices of G and L(G) are given as follows: Let

$E_2{}^1 = \{e \in E(G)\ /\ e = uv\ , u, v \in V_2(G),\quad e_G(u) = 12, e_G(v) = 12\}$

For $e \in E_2{}^1, e_{L(G)}(e) = 15$ and $|E_2{}^1| = 2$

$E_2{}^2 = \{e \in E(G)\ /\ e = uv\ , u, v \in V_2(G),\quad e_G(u) = 11, e_G(v) = 12\}$

For $e \in E_2{}^2, e_{L(G)}(e) = 14$ and $|E_2{}^2| = 4$ ....

$E_2{}^{13} = \{e \in E(G)\ /\ e = uv\ , u, v \in V_2(G),\quad e_G(u) = 7, e_G(v) = 7\}$

For $e \in E_2{}^{13}, e_{L(G)}(e) = 9$ and $|E_2{}^{13}| = 2$. Hence

$$\text{(ii)(a) } B\prod_1 E(H_6) = \prod_{ue}[e_G(u) + e_{L(G)}(e)]$$

$$B\prod_1 E(G) = \prod_{ue}[e_{H_6}(u) + e_{L(H_6)}(e)]$$

$$= \prod_{e=uv\in E_2(G)} [e_G(u) + e_{L(G)}(e)][e_G(v) + e_{L(G)}(e)]$$

$$= [(12+15)^2(12+15)^2] \times [(11+14)^4 + (12+14)^4] \times [(11+14)^2(11+14)^2] \times [(9+12)^4(10+12)^4] \times [(11+14)^2(10+14)^2] \times [(11+13)^2(10+13)^2] \times [(8+10)^2(9+10)^2] \times [(7+10)^2(8+10)^2] \times [(7+9)^2(7+9)^2] \times [(7+9)^2(8+9)^2] \times [(9+12)^2(9+12)^2] \times [(9+11)^4(8+11)^4] \times [(7+10)(7+10)] = 7.4408 \times e^{15}$$

$$\text{(b) } B\prod_2 E(H_6) = \prod_{ue}[e_G(u)e_{L(G)}(e)]$$

$$B\prod_2 E(G) = \prod_{ue}[e_{H_6}(u)e_{L(H_6)}(e)]$$

$$= \prod_{e=uv\in E_2(G)} [e_G(u)e_{L(G)}(e)][e_G(v)e_{L(G)}(e)] =$$
$$[(12\times 15)^2(12\times 15)^2]\times[(11\times 14)^4(12\times 14)^4]$$
$$\times[(11\times 14)^2(11\times 14)^2]\times[(9\times 12)^4(10\times 12)^4]$$
$$\times[(11\times 14)^2(10\times 14)^2]\times[(11\times 13)^2(10\times 13)^2]$$
$$\times[(8+10)^2(9+10)^2]\times[(7+10)^2(8+10)^2]$$
$$\times[(7+9)^2(7+9)^2]\times[(7+9)^2(8+9)^2]$$
$$\times[(9\times 12)^2(9\times 12)^2]\times[(9+11)^4(8+11)^4]$$
$$\times[(7+10)(7+10)] = 5.8852\times e^{252}$$

$$\text{(c) } HB\prod_1 E(H_6) = \prod_{ue}[e_G(u)+e_{L(G)}(e)]^2$$
$$HB\prod_1 E(G) = \prod_{ue}[e_{H_6}(u)+e_{L(H_6)}(e)]^2$$
$$= \prod_{e=uv\in E_2(G)} [e_G(u)+e_{L(G)}(e)]^2[e_G(v)+e_{L(G)}(e)]^2$$
$$= [(12+15)^4(12+15)^4]\times[(11+14)^8+(12+14)^8]$$
$$\times[(11+14)^4(11+14)^4\times[(9+12)^8(10+12)^8]]$$
$$\times[(11+14)^4(10+14)^4]\times[(11+13)^4(10+13)^4]$$
$$\times[(8+10)^4(9+10)^4]\times[(7+10)^4(8+10)^4]$$
$$\times[(7+9)^4(7+9)^4]\times[(7+9)^4(8+9)^4]$$
$$\times[(9+12)^4(9+12)^4]\times[(9+11)^8(8+11)^8]$$
$$\times[(7+10)^2(7+10)^2] = 3.2309\times e^{77}$$

$$\text{(d) } HB\prod_2 E(H_6) = \prod_{ue}[e_G(u)e_{L(G)}(e)]^2$$
$$HB\prod_2 E(G) = \prod_{ue}[e_{H_6}(u)e_{L(H_6)}(e)]^2$$
$$= \prod_{e=uv\in E_2(G)} [e_G(u)e_{L(G)}(e)]^2[e_G(v)e_{L(G)}(e)]^2$$
$$= [(12\times 15)^4(12\times 15)^4]\times[(11\times 14)^8+(12\times 14)^8]$$
$$\times[(11\times 14)^4(11\times 14)^4]\times[(9\times 12)^8(10\times 12)^8]$$
$$\times[(11\times 14)^4(10\times 14)^4]\times[(11\times 13)^4(10\times 13)^4]$$
$$\times[(8\times 10)^4(9\times 10)^4]\times[(7\times 10)^4(8\times 10)^4]$$
$$\times[(7\times 9)^4(7\times 9)^4]\times[(7\times 9)^4(8\times 9)^4]$$
$$\times[(9\times 12)^4(9\times 12)^4]\times[(9\times 11)^8(8\times 11)^8]$$
$$\times[(7\times 10)^2(7\times 10)^2] = 1.1063\times e^{123}$$

Consider the Zig-zag chain Z_6 as in the diagram

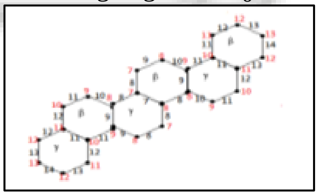


Fig. 9: A Zig-zag chain Z_6

Also the number of edges with eccentricities of end vertices of G and L(G) are given as follows: Let

$E_3^{\,1} = \{e \in E(G)\ /\ e = uv\ , u, v \in V_6(G),\quad e_G(u) = 12, e_G(v) = 13\}$

For $e \in E_3^{\,1}, e_{L(G)}(e) = 13$ and $|E_3^{\,1}| = 2$

$E_3^{\,2} = \{e \in E(G)\ /\ e = uv\ , u, v \in V_6(G),\quad e_G(u) = 12, e_G(v) = 13\}$

For $e \in E_3^{\,2}, e_{L(G)}(e) = 14$ and $|E_3^{\,2}| = 1$ ...

$E_3^{\,14} = \{e \in E(G)\ /\ e = uv\ , u, v \in V_6(G), e_G(u) = 8, e_G(v) = 8\}$

For $e \in E_3^{\,14}, e_{L(G)}(e) = 8$ and $|E_3^{\,14}| = 1$

$$\text{(iii)(a) } B\prod_1 E(Z_6) = \prod_{ue}[e_G(u)+e_{L(G)}(e)]$$
$$B\prod_1 E(G) = \prod_{ue}[e_{Z_6}(u)+e_{L(Z_6)}(e)] =$$
$$\prod_{e=uv\in E_3(G)} [e_G(u)+e_{L(G)}(e)][e_G(v)+e_{L(G)}(e)] =$$
$$[(12+13)^2(13+13)^2]\times[(13+14)(12+14)]$$
$$\times[(11+12)^2(12+12)^2]\times[(11+11)^2(10+11)^2]$$
$$\times[(11+13)^2(12+13)^2]\times[(11+12)^4(10+12)^4]$$
$$\times[(9+11)^4(10+11)^4]\times[(9+10)^3(8+10)^3]$$
$$\times[(7+8)^4(8+8)]^4\times[(7+9)(8+9)]\times[(7+7)(7+8)]$$
$$\times[(7+8)(7+8)]\times[(8+8)(8+8)]$$
$$\times[(8+9)^3(9+9)^3] = 1.0439\times e^{155}$$

$$\text{(b) } B\prod_2 E(Z_6) = \prod_{ue}[e_G(u)e_{L(G)}(e)]$$
$$B\prod_2 E(G) = \prod_{ue}[e_{Z_6}(u)+e_{L(Z_6)}(e)] =$$
$$\prod\nolimits_{e=uv\in E_3(G)}[e_G(u)e_{L(G)}(e)][e_G(v)e_{L(G)}(e)] =$$
$$[(12\times 13)^2(13\times 13)^2]\times[(13\times 14)(12\times 14)]$$
$$\times[(11\times 13)^2(12\times 13)^2]\times[(11\times 12)^4(10\times 12)^4]$$
$$\times[(11\times 11)^2(10\times 11)^2]\times[(11\times 12)^2(12\times 12)^2]$$
$$\times[(9\times 11)^4(10\times 11)^4]\times[(9\times 10)^3(8\times 10)^3]$$
$$\times[(7\times 8)^4(8\times 8)]^4\times[(7\times 9)(8\times 9)]\times[(7\times 7)(7\times 8)]$$
$$\times[(7\times 8)(7\times 8)]\times[(8\times 8)(8\times 8)]$$
$$\times[(8\times 9)^3(9\times 9)^3] = 1.2056\times e^{198}$$

$$\text{(c) } HB\prod_1 E(Z_6) = \prod_{ue}[e_G(u)+e_{L(G)}(e)]^2 =$$
$$HB\prod_1 E(G) = \prod_{ue}[e_{Z_6}(u)+e_{L(Z_6)}(e)]^2 =$$
$$\prod\nolimits_{e=uv\in E_3(G)}[e_G(u)+e_{L(G)}(e)]^2[e_G(v)+e_{L(G)}(e)]^2 =$$
$$[(12+13)^4(13+13)^4]\times[(13+14)^2(12+14)^2]$$
$$\times[(11+13)^4(12+13)^4]\times[(11+12)^8(10+12)^8]$$
$$\times[(11+11)^4(10+11)^4]\times[(11+12)^4(12+12)^4]$$
$$\times[(9+11)^8(10+11)^8]\times[(9+10)^6(8+10)^6]$$
$$\times[(7+8)^8(8+8)]^8\times[(7+9)^2(8+9)^2]$$
$$\times[(7+7)^2(7+8)^2]\times[(8+9)^6(9+9)^6]$$
$$\times[(7+8)^2(7+8)^2]\times[(8+8)^2(8+8)^2] = 1.2568\times e^{175}$$

$$\text{(d) } HB\prod_2 E(Z_6) = \prod_{ue}[e_G(u)e_{L(G)}(e)]^2$$
$$HB\prod_2 E(G) = \prod_{ue}[e_{Z_6}(u)e_{L(Z_6)}(e)]^2$$
$$= \prod_{e=uv\in E_3(G)} [e_G(u)e_{L(G)}(e)]^2[e_G(v)e_{L(G)}(e)]^2$$
$$= [(12\times 13)^4(13\times 13)^4]\times[(13\times 14)^2(12\times 14)^2]$$
$$\times[(11\times 12)^4(12\times 12)^4]$$
$$\times[(11\times 13)^4(12\times 13)^4]\times[(11\times 12)^8(10\times 12)^8]$$
$$\times[(9\times 11)^8(10\times 11)^8]\times[(9\times 10)^6(8\times 10)^6]$$
$$\times[(8\times 9)^6(9\times 9)^6]\times[(11\times 11)^4(10\times 11)^4]$$
$$\times[(7\times 8)^8(8\times 8)]^8\times[(7\times 9)^2(8\times 9)^2]$$
$$\times[(7\times 8)^2(7\times 8)^2]\times[(8\times 8)^2(8+8)^2]$$
$$\times[(7\times 7)^2(7\times 8)^2] = 1.2238\times e^{246}$$

D. *Theorem 2.4*

If L_n is a linear chain with n hexagons, then

$$\text{(i)(a)} B\prod_1 E(L_n)$$
$$= \begin{cases} 2^{16}(n+2)^8(n+1)^8\times\ldots\times 2^8(4n-1)^4(4n+1)^4(2n+1)^4, & n \text{ is odd} \\ (2n+1)^2(n+4)^2(n+5)^2\times\ldots\times(4n+3)^4(4n+4)^4(4n+5)^4, & n \text{ is even}\end{cases}$$

$$\text{(b)} B\prod_1 E(L_n)$$
$$= \begin{cases} (n+1)^{16}(n+2)^{20}(n+3)^4\times\ldots\times 4^8(2n+1)^{16}(2n+1)^4, & n \text{ is odd} \\ (n+1)^5(n+2)^6(n+3)^8\times\ldots\times 2^4n^4(2n+1)^8(2n+3)^8(2n+4)^8, & n \text{ is even}\end{cases}$$

$$\text{(c)} HB\prod_1 E(L_n)$$
$$= \begin{cases} 2^{24}(n+1)^{16}(n+2)^{32}\times\ldots\times 2^{16}n^8(4n+1)^8(4n+2)^8(4n+3)^8, & n \text{ is odd} \\ (2n+2)^{16}(2n+4)^8\times\ldots\times(4n+1)^8(4n+2)^8(4n+3)^8, & n \text{ is even}\end{cases}$$

$$\text{(d)} HB\prod_2 E(L_n)$$
$$= \begin{cases} (n+1)^{32}(n+2)^{16}\times\ldots\times 2^8n^8(2n+1)^{32}(2n+2)^8, & n \text{ is odd} \\ (n+1)^{12}(n+2)^{12}\times\ldots\times 2^8n^8(2n+1)^{16}(2n+3)^{16}, & n \text{ is even}\end{cases}$$

1) *Proof*

a) Case 1: Suppose n is odd:Consider the Linear chain L_n as in the diagram

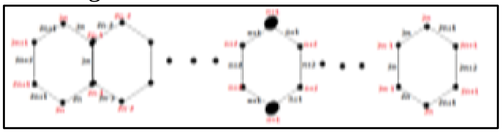


Fig. 10: Linear chainL_n , n is odd.

Let V_n be the vertex set in L_n, and E_n be the edge set in L_n, then $|V_n| = 8n - 2$ and $|E_n| = 10n - 4$.

Also the number of edges with eccentricities of end vertices are given as follows: Let

$E_n^1 = \{e \in E(G) \;/\; e = uv \;, u, v \in V_n(G), \quad e_G(u) = n + 1, e_G(v) = n + 2\}$

For $e \in E_n^1, e_{L(G)}(e) = n + 1$ and $|E_n^1| = 4$

$E_n^2 = \{e \in E(G) \;/\; e = uv \;, u, v \in V_n(G), \quad e_G(u) = n + 2, e_G(v) = n + 2\}$

For $e \in E_n^2, e_{L(G)}(e) = n + 2$ and $|E_n^2| = 2$ ...

$E_n^6 = \{e \in E(G) \;/\; e = uv \;, u, v \in V_n(G), e_G(u) = 2n + 1, e_G(v) = 2n + 1\}$

For $e \in E_n^6, e_{L(G)}(e) = 2n + 1$ and $|E_n^6| = 2$

$$(i)(a)\; B\prod_1 E(L_n) = \prod_{ue}[e_G(u) + e_{L(G)}(e)]$$
$$B\prod_1 E(G) = \prod_{ue}[e_{L_n}(u) + e_{L(L_n)}(e)]$$
$$= [((n+1)+(n+1))^4((n+1)+(n+1))^4]$$
$$\times[((n+2)+(n+2))^2((n+2)+(n+2))^2] \times \ldots$$
$$\times[(2n+2n+1)^4((2n+1)+(2n+1))^4]$$
$$\times[((2n+1)+(2n+2))^2((2n+1)+(2n+2))^2]$$
$$= 2^{16}(n+2)^8(n+1)^8 \times \ldots$$
$$\times\, 2^8(4n-1)^4(4n+1)^4(2n+1)^4$$

$$(i)(a)\; B\prod_1 E(L_n) = \prod_{ue}[e_G(u) + e_{L(G)}(e)]$$
$$B\prod_1 E(G) = \prod_{ue}[e_{L_n}(u) + e_{L(L_n)}(e)]$$
$$= [((n+1)+(n+1))^4((n+1)+(n+1))^4]$$
$$\times[((n+2)+(n+2))^2((n+2)+(n+2))^2] \times \ldots$$
$$\times[(2n+2n+1)^4((2n+1)+(2n+1))^4]$$
$$\times[((2n+1)+(2n+2))^2((2n+1)+(2n+2))^2]$$
$$= 2^{16}(n+2)^8(n+1)^8 \times \ldots$$
$$\times\, 2^8(4n-1)^4(4n+1)^4(2n+1)^4$$

$$(b)\; B\prod_2 E(L_n) = \prod_{ue}[e_G(u) + e_{L(G)}(e)]$$
$$B\prod_2 E(G) = \prod_{ue}[e_{L_n}(u) + e_{L(L_n)}(e)] =$$
$$[((n+1)(n+1))^4((n+1)(n+1))^4] \times \ldots \times$$
$$\times[((n+2)(n+2))^2((n+2)(n+2))^2]$$
$$[((2n+1)\times(2n+2))^2((2n+1)\times(2n+2))^2]$$
$$= (n+1)^{16}(n+2)^8 \times \ldots \times 4^8(2n+1)^4(2n+2)^4$$

$$(c)\; HB\prod_1 E(L_n) = \prod_{ue}[e_G(u) + e_{L(G)}(e)]^2$$
$$HB\prod_1 E(G) = \prod_{ue}[e_{L_n}(u) + e_{L(L_n)}(e)]^2 =$$
$$[((n+1)+(n+1))^8((n+1)+(n+1))^8]$$
$$\times[((n+2)+(n+2))^4((n+2)+(n+2))^4] \times \ldots \times$$
$$[((2n+1)+(2n+2))^4((2n+1)+(2n+2))^4] \times$$
$$[(2n+2n+1)^8((2n+1)+(2n+1))^8] =$$
$$(2n+2)^{16}(2n+4)^8 \times \ldots \times (4n+1)^8(4n+2)^8(4n+3)^8$$

$$(d)\; HB\prod_2 E(L_n) = \prod_{ue}[e_G(u)e_{L(G)}(e)]^2 =$$
$$HB\prod_2 E(G) = \prod_{ue}[e_{L_n}(u)e_{L(L_n)}(e)]^2 =$$
$$[((n+1)(n+1))^8((n+1)(n+1))^8]$$
$$\times[((n+2)(n+2))^4((n+2)(n+2))^4]$$
$$\times \ldots \times [(2n \times 2n+1)^8((2n+1)\times(2n+1))^8]$$
$$\times[((2n+1)\times(2n+2))^4((2n+1)\times(2n+2))^4]$$
$$= (n+1)^{32}(n+2)^{16} \times \ldots \times 2^8n^8(2n+1)^{32}(2n+2)^8$$

b) Case 2: Suppose n is even:

Consider the Linear chain L_n as in the diagram

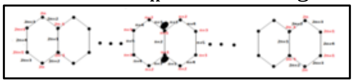


Fig. 11: Linear chain L_n , n is even.

Let V_n be the vertex set in L_n, and E_n be the edge set in L_n, then $|V_n| = 8n - 2$ and $|E_n| = 10n - 4$.

Also the number of edges with eccentricities of end vertices are given as follows: Let

$E_n^1 = \{e \in E(G) \;/\; e = uv \;, u, v \in V_n(G), \quad e_G(u) = n + 1, e_G(v) = n + 1\}$

For $e \in E_n^1, e_{L(G)}(e) = n + 2$ and $|E_n^1| = 1$

$E_n^2 = \{e \in E(G) \;/\; e = uv \;, u, v \in V_n(G), \quad e_G(u) = n + 1, e_G(v) = n + 2\}$

For $e \in E_n^2, e_{L(G)}(e) = n + 3$ and $|E_n^2| = 4$ ...

$E_n^7 = \{e \in E(G) \;/\; e = uv \;, u, v \in V_n(G), e_G(u) = 2n + 1, e_G(v) = 2n + 1\}$

For $e \in E_n^7, e_{L(G)}(e) = 2n + 4$ and $|E_n^7| = 2$

$$(i)(a)\; B\prod_1 E(L_n) = \prod_{ue}[e_G(u) + e_{L(G)}(e)]$$
$$B\prod_1 E(G) = \prod_{ue}[e_{L_n}(u) + e_{L(L_n)}(e)] =$$
$$[((n+1)+(n+2))((n+1)+(n+2))] \times$$
$$[((n+1)+(n+3))^4((n+2)+(n+3))^4]$$
$$\times \ldots \times [(2n+2n+3)^4((2n+1)+(2n+3))^4]$$
$$\times[((2n+1)+(2n+4))^2 \times ((2n+1)+(2n+4))^2]$$
$$= (n+1)^5(n+2)^6(n+3)^8 \times \ldots$$
$$\times\, 2^4n^4(2n+1)^8(2n+3)^8(2n+4)^8$$

$$(b)\; B\prod_2 E(L_n) = \prod_{ue}[e_G(u)e_{L(G)}(e)]$$
$$B\prod_2 E(G) = \prod_{ue}[e_{L_n}(u)e_{L(L_n)}(e)] =$$
$$[((n+1)(n+2))((n+1)(n+2))]$$
$$\times[((n+1)(n+3))^4((n+2)(n+3))^4]$$
$$\times \ldots \times [(2n \times 2n+3)^4((2n+1)(2n+3))^4]$$
$$\times[((2n+1)(2n+4))^2 \times ((2n+1)(2n+4))^2]$$
$$= (n+1)^5(n+2)^6(n+3)^8 \times \ldots$$
$$\times\, 2^4n^4(2n+1)^8(2n+3)^8(2n+4)^8$$

$$(c)\; HB\prod_1 E(L_n) = \prod_{ue}[e_G(u) + e_{L(G)}(e)]^2 =$$
$$HB\prod_1 E(G) = \prod_{ue}[e_{L_n}(u) + e_{L(L_n)}(e)]^2 =$$
$$[((n+1)+(n+2))^2((n+1)+(n+2))^2]$$
$$\times[((n+1)+(n+3))^8((n+2)+(n+3))^8]$$
$$\times[((n+2)+(n+4))^8((n+3)+(n+4))^8] \times \ldots \times$$
$$[(2n+2n+2)^8(2n-1+2n+2)^8]$$
$$\times[(2n+2n+3)^8((2n+1)+(2n+3))^8]$$
$$\times[((2n+1)+(2n+4))^4 \times ((2n+1)+(2n+4))^4]$$
$$\times[((n+3)+(n+5))^8((n+3)+(n+5))^8] =$$
$$(2n+2)^{16}(2n+4)^8 \times \ldots \times (4n+1)^8(4n+2)^8(4n+3)^8$$

$$(d)\; HB\prod_2 E(L_n) = \prod_{ue}[e_G(u)e_{L(G)}(e)]^2 =$$
$$HB\prod_2 E(G) = \prod_{ue}[e_{L_n}(u)e_{L(L_n)}(e)]^2 =$$
$$[((n+1)(n+2))^2((n+1)(n+2))^2]$$

$$\times\left[\left((n+1)(n+3)\right)^8\left((n+2)(n+3)\right)^8\right]$$
$$\times\ldots\times\left[(2n\times 2n+2)^8\left((2n-1)\times(2n+2)\right)^8\right]$$
$$\times\left[(2n\times 2n+3)^8\left((2n+1)(2n+3)\right)^8\right]$$
$$\times\left[\left((2n+1)(2n+4)\right)^4\times\left((2n+1)(2n+4)\right)^4\right]$$
$$=(n+1)^{12}(n+2)^{12}\times\ldots\times 2^8n^8(2n+1)^{16}(2n+3)^{16}$$

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