A Study on Divisor Cordial Graphs

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Abstract— In this chapter, we introduce a new concept called divisor cordial labeling. A divisor cordial labeling of a graph G with vertex set V is a bijection f from V to {1, 2, . . . , | V |} such that if each edge uv is assigned the label 1 if f(u) divides f(v) or f(v) divides f(u) and 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1.In this chapter, we have proved that the standard graphs such as path, cycle, wheel, star and some complete bipartite graphs are divisor cordial. We have also proved that complete graph is not divisor cordial. We show that the special graphs G∗K1,n , G∗K2,n and G∗K3,n are divisor cordial.

I. Introduction

Number Theory is a fascinating subject in mathematics. It has so many interesting concepts. The concepts of primality and divisibility play an important role in Number Theory. The origin of graph labeling problem dates back to the thirteenth century when the Chinese Mathematician Yang Hui and others studied the labeling of geometric figures, which are now classified as plane graphs. The efforts to find solutions to many practical problems in real life situations have also led to the development of several graph labeling methods - graceful, harmonious, prime, divisor, magic, antimagic, cordial, product cordial, prime cordial etc.

Various labeling methods also generate so many new types of graphs. Labeling graphs are becoming an increasingly useful family of mathematical models for a broad range of applications like designing good radar type codes with auto correlation properties, determining ambiguities in X-ray crystallographic analysis, formulating a communication network addressing system, determining optimal circuit layout and database management etc. A systematic presentation of diverse applications of graph labeling is given in [2] and [3]. Graph labeling [8] is a strong communication between Number theory [6] and structure of graphs [1]. By combining the divisibility concept in Number theory and Cordial labeling concept in Graph labeling, we introduce a new concept called 13 divisor cordial labeling.A vertex labeling [8, 10] of a graph G is an assignment f of labels to the vertices of G that induces for each edge uv a label depending on the vertex label f(u) and f(v). The two best known labeling methods are called graceful and harmonious labelings. Cordial labeling is a variation of both graceful and harmonious labelings [2, 4, 8, 9].

The cordial labeling technique was introduced by Cahit [7] in which he investigated several results on this concept. After this many labeling techniques are also introduced with minor changes in cordial labeling. The product cordial labeling, total product cordial labeling and prime cordial labeling are some of them. The present work is focused on divisor cordial labeling.

Many labeling schemes have been introduced so far and they are explored as well by many researchers. Graph labelings have enormous applications within mathematics as well as to several areas of computer science and communication networks. Various applications of graph labeling are reported in the work of Yegnanaryanan and Vaidhyanathan [1]. For a dynamic survey on various graph labeling problems along with an extensive bibliography we refer to Gallian [2].

II. Preliminary

A. Definition

Let a and b be two integers. If a divides b means that there is a positive integer k such that b = ka. It is denoted by a | b. If a does not divide b, then we denote a ∤ b.

1) Example:

We all know that 4872 is divisible by 2, since 2 · 2436 = 4872. Note that, in the notation of Definition, in this example we have a = 2, b = 4872, and x = 2436. Therefore we can write 2|4872. Another way of saying a|b is that a is a divisor of b, and we say a is a common divisor of b and c if a|b and a|c. Note that since any nonzero integer has a finite number of divisors, any two nonzero integers can have only a finite number of common divisors (of course, this is not true if b = c = 0).

B. Labeled Graph

In the mathematical discipline of graph theory, a graph labeling is the assignment of labels, traditionally represented by integers, to the edges or vertices, or both, of a graph.

Formally, given a graph G = (V, E), a vertex labeling is a function of V to a set of labels. A graph with such a function defined is called a vertex-labeled graph. Likewise, an edge labeling is a function of E to a set of labels. In this case, the graph is called an edge-labeled graph.

When the edge labels are members of an ordered set (e.g., the real numbers), it may be called a weighted graph.

When used without qualification, the term labeled graph generally refers to a vertex-labeled graph with all labels distinct. Such a graph may equivalently be labeled by the consecutive integers {1, …, |V|}, where |V| is the number of vertices in the graph. For many applications, the edges or vertices are given labels that are meaningful in the associated domain. For example, the edges may be assigned weights representing the "cost" of traversing between the incident vertices.

1) Example:

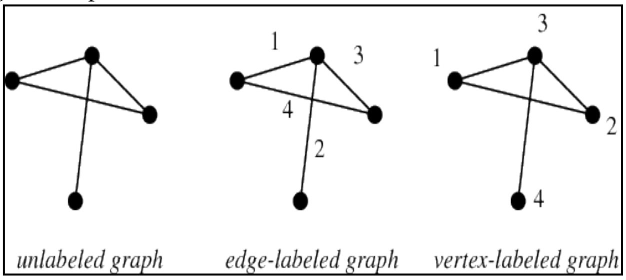

Fig. 1:

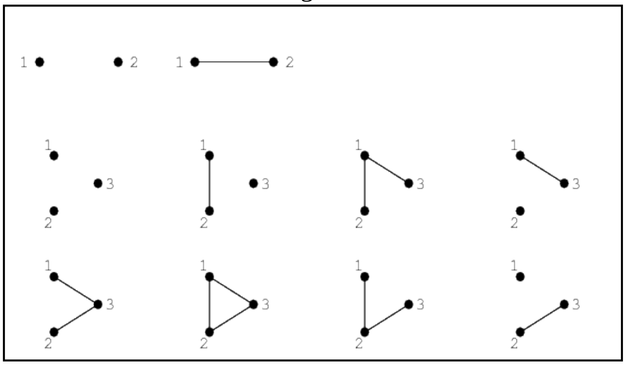

Fig. 2:

C. Divisor Cordial Labeling

Let G = (V (G), E (G)) be a simple graph and f: V (G) → {1, 2, ··· , |V (G)|} be a bijection. For each edge uv, assign the label 1 if either f (u)|f(v) or f(v)|f(u) and the label 0 otherwise. The function f is called a divisor cordial labeling if |ef(0) − ef (1)| ≤ 1.

D. Divisor Cordial Graph

A graph which admits a divisor cordial labeling is called a divisor cordial graph.

1) Definition:

Let G = (V (G), E(G)) be a simple graph with n vertices and f : V → {1,3,…,2n–1|} be a bijection. For each edge uv, assign the label 1 if either f(u)| f(v) or f(v)| f(u) and the label 0 otherwise. f is called a vertex odd divisor cordial labeling if |ef(0) − ef(1)| ≤ 1. A graph with vertex odd divisor cordial labeling is called a vertex odd divisor cordial graph.

2) Definition:

The join G = G1+G2 of graphs G1 and G2 with disjoint vertex sets V1 and V2, and edge sets E1 and E2, is the graph union G1∪G2 together with all the edges joining V1 and V2. A wheel graph Wn is defined as Cn+K1, where Cn denotes the cycle with n vertices.

3) Example:

The divisor cordial labeling technique was introduced by Varatharajan et al.[10] and they proved the following results:

- The star graph K1,n is divisor cordial.
- The complete bipartite graph K2, n is divisor cordial.
- The complete bipartite graph K3,n is divisor cordial.
- S (K1,n), the subdivision of the star K1,n is divisor cordial.
- Vaidya and Shah [11] proved that
- S(Bn, n) is a divisor cordial graph.
- DS (Bn, n) is a divisor cordial graph.
- D2 (Bn, n) is a divisor cordial graph.

1) THEOREM 1:

The path Pn is divisor cordial.

PROOF:

Let v1, v2, . . . vn be the vertices of the path Pn . Label these vertices in the following order. 1, 2, 2 2 , . . . $2^{k}1$
3, 3.2, 3.2^2 , . . . $3.2^{k}{}_2$
5, 5.2, 5.2^2 , . . . 5.2^{k3}
. . . .
Where (2m − 1)2km ≤ n and m ≥ 1, km ≥ 0.

We observe that (2m − 1)2a divides (2m − 1)2b (a < b) and (2m − 1)2ki does not divide 2m + 1. 21 In the above labeling, we see that the consecutive adjacent vertices have the labels even numbers and consecutive adjacent vertices have labels odd and even numbers contribute 1 to each edge.

Similarly, the consecutive adjacent vertices have the labels odd numbers and consecutive adjacent vertices have labels even and odd numbers contribute 0 to each edge..

Thus, ef (1) = n 2 and ef (0)= n−2 2 if n is even. and ef (1)= ef (0) = n−1 2 if n is odd. Hence, |ef (0) − ef (1)| ≤ 1.

Thus, Pn is divisor cordial.

2) THEOREM 2:

The cycle Cn is divisor cordial.

PROOF:

Let v1, v2, . . . , vn be the vertices of the cycle Cn . We follow the same labeling pattern as in the path, except by interchanging the labels of v1 and v2 .

Then, Cn is divisor cordial.

3) THEOREM 3:

The complete bipartite graph K3, n is divisor cordial.

PROOF:

Let V = X ∪ Y be the bipartition of V such that X = {x1, x2, x3} and Y= {y1, y2, . . . , yn}. Now define f(x1) = 1,f(x2) = 2, f(x3) = p where p is the largest prime number such that p ≤ n + 3 and the remaining labels to the vertices y1, y2, . . . yn .

Then e f (0) − ef (1) = 0 if n is even
1 if n is odd

Thus K3,n , is divisor cordial.

4) Example:

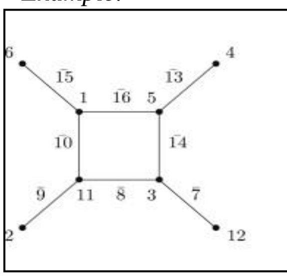

Fig. 3:

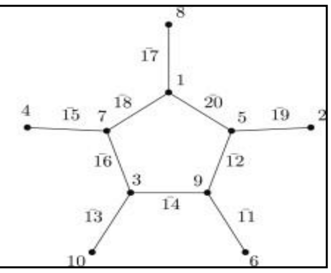

Fig. 4:

4) THEOREM 4:

Kn is not divisor cordial for n ≥ 7.

PROOF:

If possible, let there be a divisor cordial labeling f for Kn . Let v1, . . . , vn be the vertices of Kn with f(vi) = i.

First we consider vn . It contributes d(n) and (n − 1) − d(n) respectively to ef (1) and ef (0). Consequently, the contribution of vn−1 to ef (1) and ef (0) are d(n − 1) and n − 2 − d(n − 1).

Proceeding likewise, we see that vi contributes d(i) and i − 1 − d(i) to ef (1) and ef (0) respectively, for i = n, n − 1, . . . , 2. Then 31 using Result 1.37, it follows that |ef (0) − ef (1)| = 2{d(n) + . . . + d(2)} − {(n − 1) + . . . + 1} = 2{D(n) − d(1)} − {(n−1)(n−2) 2 } = 2{nlogn + n(2n-1)+Δ(n) − 1} − {(n−1)(n−2) 2 } ≥ 2 for n ≥ 7.

Thus, Kn is not divisor cordial.

5) THEOREM 5:

$C_4^{(t)}$ is divisor cordial.

PROOF:

Let $v_1^{(i)}$, $v_2^{(i)}$, $v_3^{(i)}$, $v_4^{(i)}$ (i = 1, 2, . . . , t) be the vertices of $C_4^{(t)}$ C (t) 4 with

$v_1^{(1)} = v_1^{(2)} = \cdots = v_1^{(t)} = v$ (say). It has 3t + 1 vertices and 4t edges.

Define f : V $(C_4^{(t)})$ → {1, 2, . . . , 3t+1} by

$f(v) = 1, f(v_2^{(i)}) = 3i - 1, f(v_3^{(i)}) = 3i, f(v_4^{(i)}) = 3i + 1$, for $1 \leq i \leq t$.

Since 1 divides any integer, two edges of each $C_4^{(i)}$ adjacent to v1 contribute 2 to ef (1) and since n does not divide n + 1 for n > 1, the other two edges of each $C_4^{(t)}$ contribute 2 to ef (0) for 1 ≤ i ≤ t. Hence, ef (0) = ef (1) = 2t.

Thus $C_4^{(t)}$ is divisor cordial.

6) THEOREM 6:

The wheel graph Wn = K1 + Cn−1 is divisor cordial.

PROOF:

Let v1 be the central vertex and v2, v3, . . . , vn be the vertices of Cn−1 .

Case (i): n is odd.

Label the vertices v1, v2, . . . , vn as 1, 2 . . . n. Since 1 divides all the integers and n ∤ n + 1, for n > 1, it follows that ef (0) = ef (1) = n − 1.

Case (ii): n is even.

If we label as in the case(i), then f(v2) | f(vn) and we have ef (0) = n − 2 and ef (1) = n. So we interchange the labels of vn−1 and vn . Then f(v2) ∤ f(vn) and since n ∤ n + 2 for n > 2, f(vn−1) ∤ f(vn) we have ef (0) = ef (1) = n − 1. Hence |ef (0) − ef (1)|= 0.

Thus Wn is divisor cordial.

5) Example:

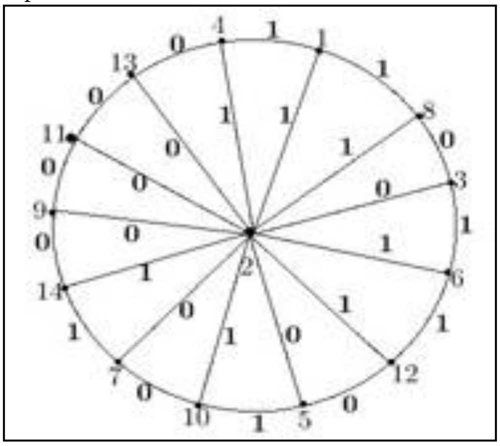

Fig. 5:

7) THEOREM 7:

Let G be any divisor cordial graph of even size. Then the graph G ∗ K1,n obtained by identifying the central vertex of K1,n with that labeled 2 in G is also divisor cordial.

PROOF:

Let q be the even size of G and let f be a divisor cordial labeling of G.

Then it follows that, ef (0)=q/2= ef (1). Let v1, v2, . . . , vn be the pendant vertices of K1,n . Extend f to G ∗ K1,n by assigning f(vi) = |V |+i (1≤ i ≤ n). In G ∗ K1,n , we see that |ef (0) − ef (1)| = 0 or 1 according as n is even or odd.

Thus, G ∗ K1,n is also divisor cordial.

8) THEOREM 8:

The graph S′(K2,m) is divisor cordial graph.

PROOF:

Let u1, u2, v1, v2, …, vm, be the vertices of K2, m.

Then u1,u2,v1,v2, …, vm, u′1, u′2, v′1, v′2, …, v′m are the vertices of S′(K2,m) and |V(G)| = 2m+4 and | E(G) | = 6m. p1 and p2 are the largest and next largest prime numbers.

Define f : V(G) → {1, 2, 3, …, 2m+4} by f(u1) = 1, f(u2) = p1, f(x′1) = 2, f(x′2) = p2, f(vi) = 2+2i, 1 ≤ i ≤ m. f(v′m) = 2m+4, Label the vertices v′1, v′2, …, v′m−1 with odd numbers from 3, 5, …, 2m+3 other than p1 and p2. Then ef(0) = ef(1) = 3m for any m. Therefore, |ef(0) − ef(1)| ≤ 1.

Hence G is divisor cordial graph.

III. CONCLUSION

In this paper, we have proved that the standard graphs such as path, cycle, wheel, star and some complete bipartite graphs are divisor cordial. We have also proved that complete graph is not divisor cordial. Also, in this paper, we introduce a new concept called divisor cordial labeling.

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