A Study on Knot Theory in Algebra

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Abstract— This paper gives an introduction to knot theory and some of its application. The preliminaries in knots and knot invariants are discussed. The elementary form of knot theory includes the transformation of unit circle into three-dimensional space. A parity result is discussed in this article that classifies certain relationships of a model with knot.
Key words: Knot Theory, Algebra

I. INTRODUCTION

Knot theory is a branch of algebraic topology where one studies what is known as the placement problem, or the embedding of one topological space into another. The simplest form of knot theory involves the embedding of the unit circle into three-dimensional space. For the three-dimensional Euclidean space R^3.

Two or more knots together are called a link. Thus a mathematical knot is somewhat different from the usual idea of a knot, that is a piece of string with free ends. The knots studied in knot theory are always considered to be closed loop.

Two knots or link are considered equivalent if one can be smoothly deformed into the other, or equivalently, if there exists a homeomorphism on R^3 which maps the image of the first knot onto the second. Cutting the knot or allowing it to pass through itself are not permitted. In generally it is very difficult problem to decide if two given knots are equivalent, and much of knot theory is devoted to developing techniques to aid in answering this question. Knots that are equivalent to polygonal paths in three-dimensional are called tame. All other knots are known as wild. Most of knot theory concerns only tame knots, and these are the only knots examined here. Knots that are equivalent to the unit circle are considered to be unknotted are trivial.

II. MATERIALS AND METHODOLOGY

A. Semi Group

A semi group is an algebraic structure consisting of a set together with an associative binary operation. The binary operation of a semi group is most often denoted multiplicatively: X.Y, or simply XY, denotes the result of applying the semi group operation to the ordered pair (X, Y).

B. Commutative Semigroup

In mathematics, a semigroup is a nonempty set together with an associative binary operation. Thus the class of commutative semigroup consists of all those semigroups in which the binary operation satisfies the commutative property that ab = ba for all elements a and b in the semigroup.

C. Generator

A cyclic group is a group that is generated by a single element. That means that there exists an element g, say, such that every other element of the group can be written as a power of g. This element g is the generator of the group.

D. Trivial Knot

The unknot in the mathematical theory of knots. Intuitively, the unknot is a closed loop of rope without a knot in it.

A knot theorist would describe the unknot as an image of any embedding that can be deformed, i.e. ambient isotope, to the standard unknot, i.e. the embedding of the circle as a geometrically round circle. The unknot is also called the trivial knot. An unknot is the identity element with respect to the knot sum operation.

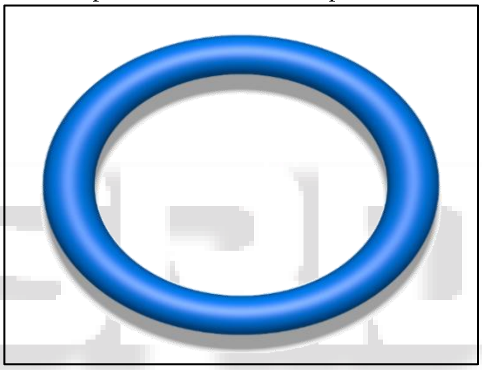

Fig. 3: Unknot

The unknotting number u(K) of a knot K is the minimum, over all diagrams D of K, of the minimal number of crossing changes required to turn D into a diagram of the unknot.

It seems intuitively clear that any diagram can be changed into a diagram of the unknot simply by switching some of the crossings. The unknotting number is then the minimal number of such changes necessary. But we really should give a proof of this fact, because otherwise we don't even know that the unknotting number is always finite!

Experience shows that if one draws a knot diagram by hand, only lifting the pen from the page when one is about to hit the line already drawn, the result is an unknot. All we do is formalise this idea.

E. Prime Knot

In knot theory, a prime knot or prime link is a knot is, in a certain sense, it is a non-trivial knot which cannot be written as the knot sum of two non-trivial knots. Knots that are not prime are said to be composite knots or composite links. It can be a nontrivial problem to determine whether a given knot is prime or not.

A family of examples of prime knots are the tours knots. These are formed by wrapping a circle around a tour p times in one direction and q times in the other, where p and q are coprime integers.

The simplest prime knot is the trefoil with three crossings. The trefoil is actually a (2, 3)-torus knot. The figure-eight knot, with four crossings, is the simplest non-torus knot.

For any positive integer n, there are a finite number of prime knots with n crossings. The first few values (sequence A002863 in the OEIS) are given in the following figure.

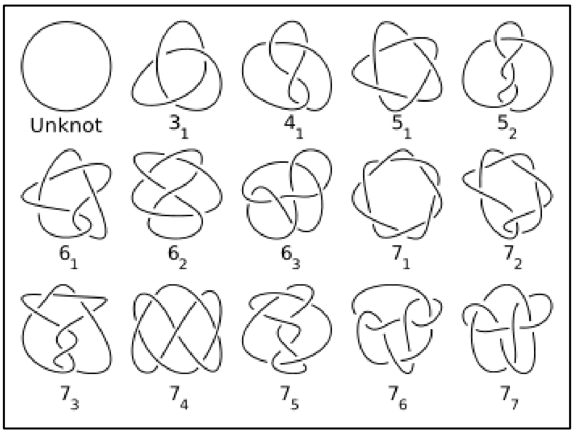

Fig. 2: A Chart of all Prime Knots with Seven or Fewer Crossings, not including Mirror-Images. (The Unknot is not Considered Prime.)

F. *Linking Number*

The linking number is a numerical invariant that describes the linking of two closed curves in three – dimensional space. Intuitively, the linking number represents the number of times that each curve winds around the other.

The linking number is always an integer, but may be positive or negative depending on the orientation of the two curves.

The linking number was introduced by Gauss in the form of the linking integral. It is an important object of study in knot theory, algebraic topology, and differential geometry, and has numerous applications in mathematics and science, including quantum mechanics, electromagnetism, and study of DNA supercoiling.

Let D be a diagram of an oriented link. Then the total linking number Lk(D) is obtained by taking half the sum, over all crossings, of contributions from each given by if the two arcs involved in the crossing belong to different components of the link, and 0 if the belong to the same one.

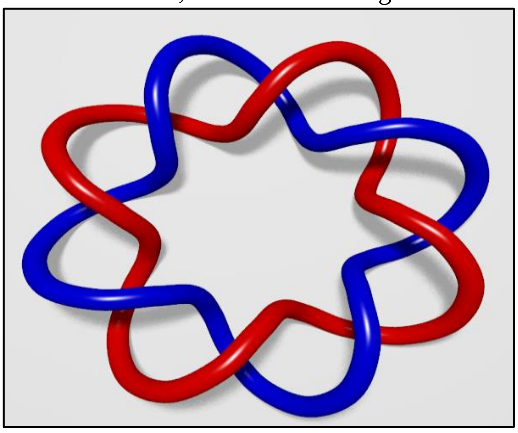

Fig. 4: The Two Curves of this (2,8)-Torus Link have Linking Number Four

G. *Seifert Surfaces*

A Seifert surface (named after German mathematician Herbert Seifert) is a surface whose boundary is a given knot or link.

Specifically, Let L be a tame oriented knot or link in Euclidean 3-Space. A Seifert surface is a compact, connected, oriented surface S embedded in 3-space whose boundary is L such that the orientation on L is just the induced orientation from S, and every connected component of S has non-empty boundary.

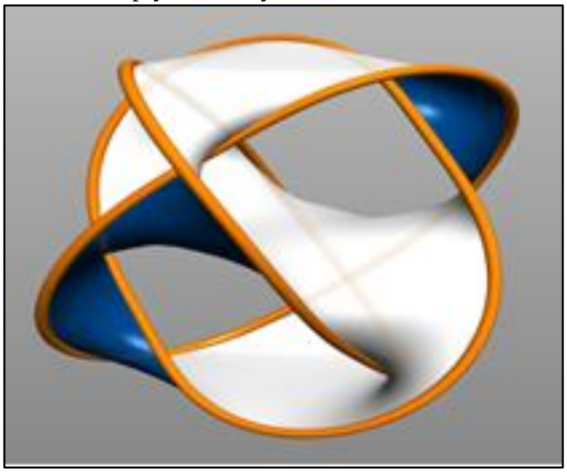

Fig. 1: Example of Seifert Surface

The standard Mobius strip has the unknot for a boundary but is not considered to be a Seifert surface for the unknot because it is not orientable.

The “checkerboard” colouring of the usual minimal crossing projection of the trefoil knot gives a Mobius strip with three half twists. As with the previous example, this is not a Seifert surface as it is not orientable. Applying Seifert’s algorithm to this diagram, as expected, does produced a Seifert surface; in this case, it is a punctured tour of genus g = 1, and this Seifert matrix is a surface.

H. *Minimal Genus Seifert Surfaces*

The genus of a knot k is the invariant defined by the minimal genus g of a Seifert for k. An unknot which is by definition, the boundary of a disc has genus zero. Moreover, the unknot is the only knot with genus zero.

I. *Reidemeister Moves*

Reidemeister move is any of three local move on a link diagram. Reidemeister (1927) and, independently, Alexander and Briggs (1926), demonstrated that two knot diagrams belonging to the same knot, upto planer isotopy, can be related by a sequence of the three Reidemeister moves.

III. CONCEPTS AND THEOREMS

A. *Lemma:*

The function g is additive, i.e., for any knots $K_1; K_2$ the equality holds

$$g(K_1) + g(K_2) = g(K_1 + K_2)$$

Proof:

First, let us show that

$$g(K_1 + K_2) \le g(K_1) + g(K_2)$$

Consider the Seifert surfaces F_1 and F_2 of minimal genii for the knots K_1 and K_2 without loss of generality, we can assume that these surfaces do not intersect each other. Let us connect the two by a band with respect to their orientation.

Thus we obtain a Seifert surface for the knot $K_1 + K_2$of genus $g(K_1) + g(K_2)$Thus

$$g(K_1 + K_2) \leq g(K_1) + g(K_2)$$

Now, let us show that

$$g(K_1 + K_2) \geq g(K_1) + g(K_2).$$

Consider a minimal genus Seifert surface F for the knot $K_1 + K_2$. There exists a (topological) sphere S^2, separating knots K_1 and K_2 in the connected sum $K_1 + K_2$at some points A,B.

The sphere S^2 intersects the surface F along several closed simple curves and a curve ended at points A,B. Each circle divides the sphere S^2 into two parts; one of them does not contain points of the curve AB. Without loss of generality, assume that the intersection $F \cap S^2$ consists of several closed simple curves $a_1, \ldots\ldots\ldots.., a_k$ and one curve connecting A and B. The neighbourhood of each a_i looks like a cylinder. Let us delete a small cylindrical part that contains the circle (our curve) from the cylinder and glue the remaining parts of the cylinder by small discs. If the obtained surface is not connected, let us take the part of it containing $\mathrm{K}_1 + \mathrm{K}_2$. After performing such operations to each circle, we obtain a closed surface F' containing the knot $\mathrm{K}_1 + \mathrm{K}_2$and intersecting S^2only along AB. The operations described before can only increase the number of handles. Thus, $\mathrm{g}(\mathrm{F}') \leq \mathrm{g}(\mathrm{F})$. Because F has minimal genus, we conclude that

$$\mathrm{g}(\mathrm{F}') = \mathrm{g}(\mathrm{F}) = \mathrm{g}(\mathrm{K}_1 + \mathrm{K}_2).$$

The sphere S^2 divides the surface F' into surfaces that can be treated as Seifert surfaces for K_1 and K_2. Thus,

$$\mathrm{g}(\mathrm{K}_1) + \mathrm{g}(\mathrm{K}_2) \geq \mathrm{g}(\mathrm{F}') = \mathrm{g}(\mathrm{K}_1).$$

Taking into account that we have proved the "$\leq$" inequality, we conclude that the genus is additive.

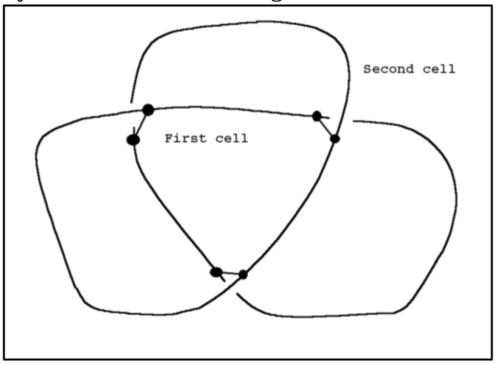

Fig. 5: Cell Decomposition of a Surface

B. Lemma

Let L and M be knots, and let K be a prime knot dividing $L\#M$. Then either K divides L or K divides M.

Proof:

Consider a knot $L\#M$ together with some plane p, intersecting it in two points and separating L from M. Since $L\#M$ is divisible by K, there exists a (topological) 2-sphere S^2that intersects the knot L#M in two points and contains the knot K (more precisely, the "Long Knot" obtained from K by stretching the ends afar) inside.

If this sphere does not intersect p, the problem would be solved. Otherwise, the sphere S^2 intersects the plane at some non-intersecting simple curves (circles). If each of these circles is unlinked with $L\#M$, they can be removed by a simple deformation. In the remaining case, they can also be removed by some deformation of the sphere because of the primitivity of K.

Thus, if the connected sum $L\#M$ is divisible by K, then either L or M is divisible by K.

1) Remark:

This basic statement of knot arithments belongs to Schubert, see

Thus we have:

1) Knot isotopy classes form a commutative semigroup related to the concatenation operation; the unit element of the semigroup is the unknot;
2) Each non-trivial knot has no inverse element;
3) Prime decomposition is unique up to permutation;
4) The number of different prime knots is denumerable.

C. Theorem

If D, $D'$are two diagrams of an oriented link L then $L_k(D) = L_k(D')$, and hence this number is an invariant $L_k(L)$, the total linking number of L.

Proof:

The two diagrams differ by a sequence of Reidemeister moves, so all we need to do is check that each of these preserves the linking number. Certainly planar isotopy preserves it. In all the other move we need only compare the local contributions from the pictures on each side, as all other crossings are common to both diagrams (Fig6). In R_1, one side has an extra crossing but it is a self-crossing, so contributes nothing extra.

In R_2, one side has two extra crossings: either the two strings involved belong to the same component (in which case both extra crossings are worth 0) or they belong to different components , in which case their contributions are equal and opposite, whatever the orientation on the strings (there are four cases).

For R_3, each of the three crossings on the left has a counterpart on the right which gives the same contribution, whatever the status of the strings involved or their orientation. Hence the sum of the three is the same on each side.

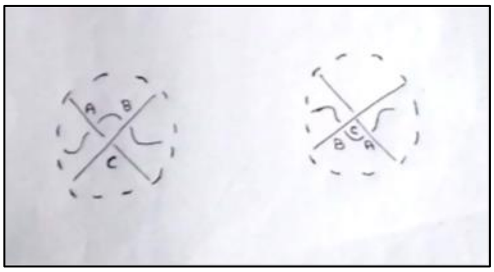

Fig. 6:

D. lemma

Any knot diagram can be changed to a diagram of the unknot by switching some of its crossings.

Proof:

Take a knot K in $\mathbb{R}^3$ with diagram (Fig7) D. Take a line L parallel to the y-axis and tangent to the knot at its left-hand side, so that the whole diagram is to the right" of this line in $\mathbb{R}^2$. Parametrise the knot in $\mathbb{R}^3$, starting over p, by a map $t \rightarrow (x(t); y(t); z(t)), for\ 0 \leq t \leq 1$. (This map will be injective except for the fact that the t = 0; t = 1 both map to a point above p).

Now make a new knot k' by gluing the image of $t \rightarrow (x(t), y(t), 1 - t)$ to a vertical arc-segment connecting its endpoints (p, 1) and (p, 0).

This knot has the same XY-projection as K (technically it's irregular, because of the vertical edge, but this is irrelevant here), but its diagram (Fig7) D' is obtained from D by changing some of the crossings: you move around k' starting from p, the monotonically descending Z-coordinate 1 - t means that you always go over the first time you reach a crossing, and under the second time. The resulting diagram (Fig8) is called the standard descending projection. But now "look along L", meaning project k' into the XZ plane , orthogonal to L;

We get a projection with no crossings, because the Z-coordinate was monotonic and the whole knot lies on one side of L. Therefore k' is an unknot, and D' is a diagram of this unknot.

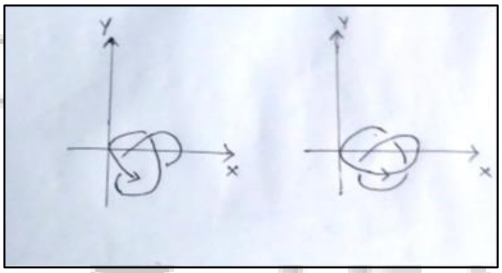

Fig. 7: D and D'

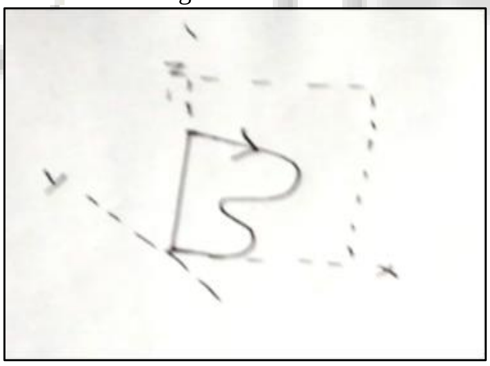

Fig. 8:

E. *Corollary*

For any knot K, $u(K) \leq \frac{c(K)}{2}$.

Proof:

Applying the above procedure to a diagram with the minimal crossing number $c(K)$, we use at most $c(K)$ crossing changes to obtain an unknot K'. If we actually take more than $\frac{c(K)}{2}$, change K instead to the unknot K_0 whose z-coordinate is t instead of $1 - t$.

This is achieved by changing exactly the crossings we didn't change to get K_0, so takes at most $\frac{c(K)}{2}$.

F. *Theorem:*

$k_{n-1} \rightarrow F(x)$,if ($b_1, b_2, \ldots\ldots, b_n$) are generators of k_{n-1} then $\{t_r(b_1), t_r(b_2), \ldots\ldots\ldots t_r(b_n)\} = \left\{1, x^2, x^4, \ldots\ldots\ldots\ldots x^{2(n-1)}, \frac{1}{x^2}, \ldots\ldots\ldots\ldots \frac{1}{x^{2(n-1)}}\right\}$

Proof:

Let us prove the theorem by induction on 'n'

For n=2, that is $A_2 = \left\{1, x^2, \frac{1}{x^2}\right\}$

Hence the result for n=2

Let us assume that the result is true foe n=k

Hence $A_k = \{t_r(b); b \epsilon K_k\}$

That is $\left\{1, x^2, x^4, \ldots\ldots\ldots\ldots x^{2(k-1)}, \frac{1}{x^2}, \ldots\ldots\ldots\ldots \frac{1}{x^{2(k-1)}}\right\}$

For n=k+1

$$A_{k+1} = \{t_r(b);\ b \epsilon K_k\}$$

1) Case 1:

Let $b \epsilon K_k$ and $(\propto_i, \propto_{k+1})\ \epsilon\ B_\pi$

where $\pi\ \epsilon\ S_n$

$e_{k+1} b = b' e_{k+1}$, $b' \epsilon\ S_n$

$$\mathcal{E}_{k+1} : K_{k+1} \rightarrow F(x)$$

$$(\mathcal{E}_{k+1}(b)) = \frac{1}{x^2}(x^2 b') = b'$$

$t_r(b) = t_r(\mathcal{E}_{k+1}(b)) = t_r(b')$ where $t_r(b') \epsilon A_k$

This is

$$t_r(b) = t_r\left\{1, x^2, x^4, \ldots\ldots\ldots\ldots x^{2(n-1)}, \frac{1}{x^2}, \ldots\ldots\ldots\ldots \frac{1}{x^{2(n-1)}}\right\}$$

2) Case 2:

Let $b \epsilon K_k$ and $(\propto_i, \propto_{k+1})\ \epsilon\ B_\pi$

where $\pi\ \epsilon\ S_n$ and $\propto_i$ is lower than $\propto_{k+1}$

$e_{k+1} b = b' e_{k+1}$, $b' \epsilon\ S_n$

$$\mathcal{E}_{k+1}(b) = \frac{1}{x^2}(b)$$

$$t_r(b) = t_r(\mathcal{E}_{k+1}(b)) = t_r(b') = t_r \frac{1}{x^2}(b')$$

This is

$$t_r(b) = \frac{1}{x^2}\left\{1, x^2, x^4, \ldots\ldots\ldots\ldots x^{2(k-1)}, \frac{1}{x^2}, \ldots\ldots\ldots\ldots \frac{1}{x^{2(k-1)}}\right\}$$

$= \left\{1, x^2, x^4, \ldots\ldots\ldots\ldots x^{2k}, \frac{1}{x^2}, \ldots\ldots\ldots\ldots \frac{1}{x^{2k}}\right\}$

3) Case 3:

Let $b\ \epsilon\ K_k$ and $(\propto_i, \propto_{k+1})\ \epsilon\ B_\pi$

where $\pi\ \epsilon\ S_n$ and $\propto_i$ is lower than $\propto_{k+1}$

$e_{k+1} b = b' e_{k+1}$, $b' \epsilon\ S_n$

$$(\mathcal{E}_{k+1}(b)) = \frac{1}{x^2}(x^4 b') = t_r(x^2 b')$$

$$t_r(b) = t_r(\mathcal{E}_{k+1}(b)) = t_r(b') = t_r(x^2 b')$$

$= x^2 t_r(b')$

$= x^2\left\{1, x^2, x^4, \ldots\ldots\ldots\ldots x^{2(k-1)}, \frac{1}{x^2}, \ldots\ldots\ldots\ldots \frac{1}{x^{2(k-1)}}\right\}$

$= \left\{1, x^2, x^4, \ldots\ldots\ldots\ldots x^{2k}, \frac{1}{x^2}, \ldots\ldots\ldots\ldots \frac{1}{x^{2k}}\right\}$

Hence the result is true for n=k+1

By the introduction hypothesis, the theorem is true for all n.

G. *Theorem*

$t_r : k_n(x) \rightarrow F(x)$, is non degenerate

Proof:

Let

$x = \sum_i \lambda_i b_i , \{b_i\}$ be the basis of $k_n : \lambda_i \epsilon F(x)$

$t_r(xy) = 0 \ \forall \, y \epsilon k_n(x)$.
In particular

$$t_r(xbj) = 0 \ \forall \, j$$

$$t_r\left(\sum_i \lambda_i b_i \, b_j\right) = 0$$

$$\sum_i \lambda_i \, t_r\left(b_i b_j\right) = 0$$

Put $K = t_r\left(b_i b_j\right)$ & Qf(x) = det(k) is a non-zero polynomial
Hence $\lambda_i = 0 \ \forall$ i, which implies X=0.

IV. CONCLUSION

This paper we have discussed about the Knot theory and some of its application. It includes some preliminaries about Knots and Knot invariants. We have given some theorem a parity result about the states in this model that classifies certain relationships of the model with Knot.

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