Performance Based Analysis of LQR Controller for Stabilizing Cart Triple Link Inverted Pendulum

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Abstract— This paper focuses on the control problem of the cart triple link inverted pendulum by LQR controller. Firstly a mathematical model for the cart triple link inverted pendulum system is presented using Lagrange equation. This model is then used to design a full-state feedback controller based on the LQR method to stabilize the system. It is then compared with an observer based LQR controller. The stability, controllability and observability are investigated and the choice of weights in LQR also discussed. The system is simulated in MATLAB environment and the simulation results establish the satisfactory performance of LQR controller in stabilizing the system.

Key words: Linear Quadratic Regulator, Observer based control, Triple Link Inverted Pendulum, Matlab, System Performance

I. Introduction

The inverted pendulum is a highly popular system studied in control labs. It is a typical example of multivariable, non linear, higher order system. It is widely used by many researchers due to its challenging features like high instability, high non linearity, non minimum phase, under actuated system. Its features enable it to be used as an experiment platform to study and test various control methods and algorithms. It is simplified version of many real world systems like walking robots, satellite launchers, missile launchers, human walking, Segway, automatic aircraft landing system, biped locomotive machines, flexible space structures and many more industrial applications. Thus it is of great significance to study the stability of the inverted pendulum in both theory and practical applications. Control of the Inverted pendulum system has been one of the challenging topics in control theory [1].The inverted pendulum system consists of a rigid rod which is joined to a cart by a bolt thus providing it rotational freedom. The triple inverted pendulum system is an extension to the inverted pendulum. It consists of three rods and a cart connected to each other by bolts thus increasing the degree of freedom. As the number of link increases the system becomes more complex.

There are various types of pendulum such as, the simple inverted pendulum, the rotary inverted pendulum, double inverted pendulum, the rotary double inverted pendulum [2].In this paper a triple link inverted pendulum is mounted on a cart. The TLIP system is a SIMO (Single input multi Output) a state space design approach is used as it well suited to the control of multiple outputs. In this paper, a continuous time linear quadratic regulator (LQR) with degree of stability is used. LQR controller is an important tool in modern control theory. For feedback system design which is adapted to time varying systems it provides an effective analysis method. The work is then extended to observer based LQR controller. The main aim of our work is to evaluate the performance of LQR controller for the cart TLIP system. The concept of stability, controllability and observability is also discussed. We will attempt to control both the pendulum's angle and cart's position. The rest of the paper deals with section 2 explain the mathematical modeling of the cart TLIP system. Section 3 is associated with designing of LQR controller with degree of stability section 4 with Observer based LQR controller section 5 with simulation and results and section 5 with conclusion.

II. Mathematical Modeling

The mathematical model of the cart TLIP system is obtained by Euler's Lagrange equation. [4], [5] The schematic of the system is shown below. The pendulum consists of three links of different lengths which are mounted on a cart., u is external action; x is displacement of cart; θ_1, θ_2, θ_3 are the angles of the lower, middle, and upper pendulum bars respectively with respect to the vertical line; m_0 is the mass of the cart; m_1, m_2, m_3 are the Centre masses of the lower, middle, upper pendulum bar respectively; L1,L2,L3 are the length of the lower, middle ,upper pendulum respectively.

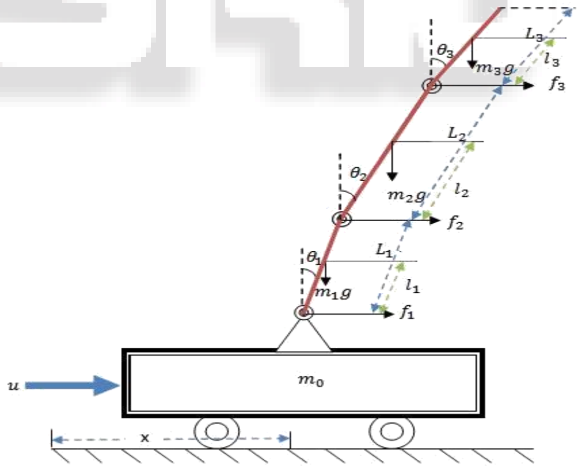


Fig. 1: Schematic of cart TLIP

The generalized Euler-Lagrange equation is given as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = -\left(\frac{\partial D}{\partial \dot{q}_i}\right) \quad (1)$$

Where, Lagrange function (Lagrangian), L = T – V
T = Total Kinetic energy of the system
V = Total Potential energy of the system
W = Work done against friction (Dissipative forces)
$q_1, q_2, \ldots, q_s$ are the generalized coordinates of the system
The mathematical model of the triple inverted pendulum is constructed based on the Lagrange equations [8]

$$M(\theta_1,\theta_2,\theta_3)\begin{bmatrix}\ddot{x}\\ \ddot{\theta}_1\\ \ddot{\theta}_2\\ \ddot{\theta}_3\end{bmatrix}+N(\theta_1,\theta_2,\theta_3,\dot{\theta}_1,\dot{\theta}_2\dot{\theta}_3)\begin{bmatrix}\dot{x}_1\\ \dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\theta}_3\end{bmatrix}=G(u,\theta_1,\theta_2,\theta_3)$$

$$G_0 = [K_s\ 0\ 0\ 0]^T$$

$$M_0 = \begin{bmatrix} k_0 & k_1 & k_2 & m_3 l_3 \\ k_1 & k_3 & k_2 L_1 & m_3 L_1 l_3 \\ k_2 & a_2 L_1 & b_2 & m_3 L_2 l_3 \\ m_3 l_3 & m_3 L_1 l_3 & m_3 L_2 l_3 & J_3 + m_3 l_3^2 \end{bmatrix}$$

$$N_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_1 g & 0 & 0 \\ 0 & 0 & k_2 g & 0 \\ 0 & 0 & 0 & m_3 l_3 g \end{bmatrix}$$

$$F_0 = \begin{bmatrix} -f_0 & 0 & 0 & 0 \\ 0 & -f_1 - f_2 & f_2 & 0 \\ 0 & f_2 & -f_2 - f_3 & f_3 \\ 0 & 0 & f_3 & -f_3 \end{bmatrix}$$

Where, the coefficients are given as:

$$\left.\begin{aligned} k_0 &= m_0 + m_1 + m_2 + m_3 \\ k_1 &= m_1 l_1 + m_2 L_1 + m_3 L_1 \\ k_2 &= m_2 l_2 + m_3 L_2 \\ k_3 &= J_1 + m_1 l_1^2 + m_2 L_1^2 + m_3 L_1^2 \\ k_4 &= J_2 + m_2 l_2^2 + m_3 L_2^2 \end{aligned}\right\} \qquad (2)$$

The linear model of the triple link inverted pendulum is represented in state-space form as follows:

$$\dot{X} = AX + BC$$
$$\dot{X} = \begin{bmatrix} 0_{4\times4} & I_{4\times4} \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} 0_{4\times1} \\ B_2 \end{bmatrix} U \qquad (3)$$
$$Y = C\,X$$

The state vector is defined by:

$$X = [x\ \ \theta_1\ \theta_2\ \theta_3 \dot{x}\ \ \dot{\theta}_1\ \dot{\theta}_2\ \ \dot{\theta}_3]^T \qquad (4)$$

The coefficient matrices of state equation (3) of cart triple inverted pendulum after putting the parameter values mentioned in APPENDIX are as follows:

$$A_{21} = \begin{bmatrix} 0 & -15.1861 & 2.7420 & -0.3023 \\ 0 & 147.1379 & -82.4455 & 9.0896 \\ 0 & -182.6242 & 194.3229 & -41.2087 \\ 0 & 37.9468 & -77.6656 & 45.8506 \end{bmatrix}$$

$$A_{22} = \begin{bmatrix} -5.2356 & 0.0136 & -0.0107 & 0.0027 \\ 17.1258 & -0.1778 & 0.2052 & -0.0822 \\ -6.8497 & 0.2967 & -0.4352 & 0.2246 \\ 1.4233 & -0.0924 & 0.2138 & -0.1355 \end{bmatrix}$$

$$B = [0\quad 0\quad 0\quad 0\quad 3.7397\quad -12.2326\quad 4.8926\quad -1.0166]^T$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

III. CHARACTERISTICS ANALYSIS

After obtaining the mathematical model of the system features, we need to analyze the stability; controllability and observability of system's in order to further understand the characteristics of the system

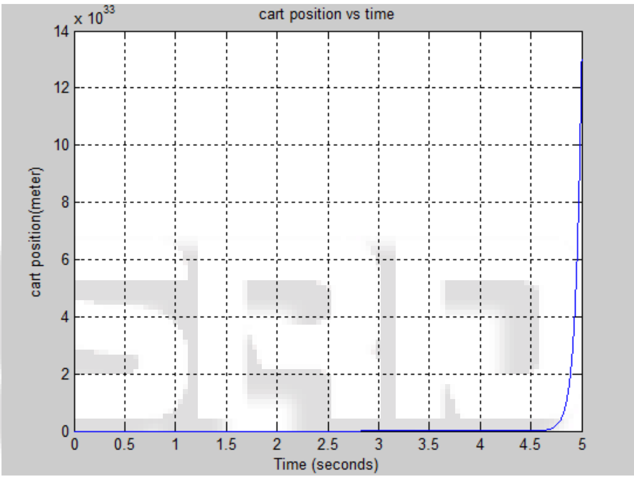
A. Stability analysis:

If the closed loop poles are all located in the left half of the s-plane, the system must be stable, otherwise the system is unstable. In MATLAB to strike a linear time invariant system, the characteristics roots can be obtained by eig(A,B)

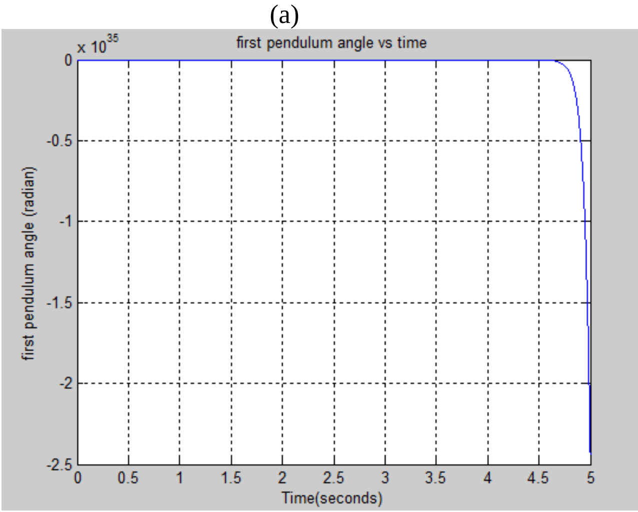
S. No.	Eigen Values of unstable system
1	0
2	16.9833
3	-18.2763
4	7.3830
5	3.8938
6	-9.0828
7	-1.9199
8	-4.9833

Table 1: Eigen values of open loop system

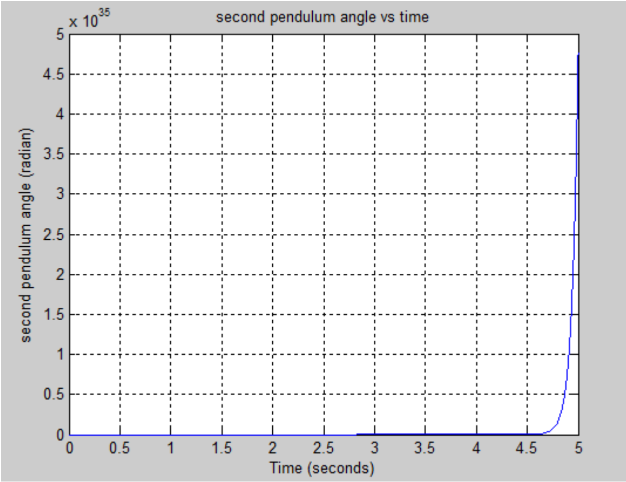
The Eigen values of the system matrix A for the system are given in Table 2. The system is unstable as it has positive Eigen values. The model is simulated in MATLAB and the step response of the open loop TLIPS are obtained as shown below



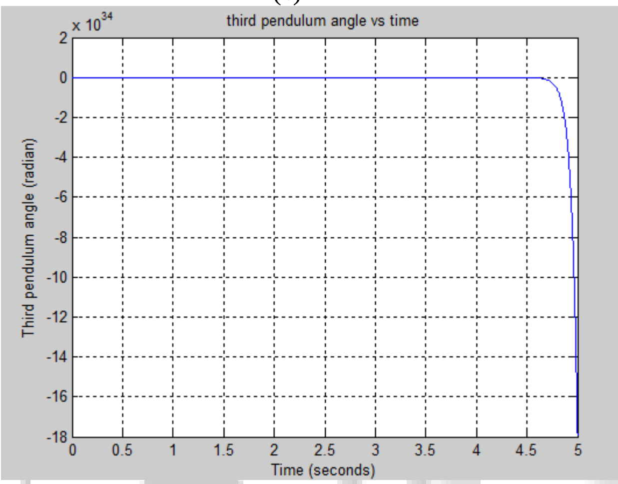
(a)



(b)



(c)



(d)

Fig. 2(a), (b),(c), (d): Step response of open loop system

From the above graph it is clear that the triple link inverted pendulum system is unstable

B. Controllability Analysis:

Linear time-invariant controllability systems necessary and sufficient condition is:

Rank (C) =n Where C is controllability matrix given by

C= [B AB A^2B……. A^{n-1}B]

The dimension of the matrix A is n. In MATLAB, the function ctrb(a,b) is used to test the controllability of matrix ,through the calculation we can see that the system is controllable.

C. Observability analysis:

Linear time-invariant observability systems necessary and sufficient condition is:

Rank (O)=n where O is observability matrix given by

O= [C CA CA^2……CA^{n-1}] ^{T}

In MATLAB, the function obsv(a,b) is used to test the observability of matrix ,through the calculation we can see that the system is observable.

IV. DESIGN OF LQR CONTROLLER

LQR is a control scheme that provides the best possible performance with respect to some given measure of performance. Our LTI plant model with directly measurable state x is given as

$$\dot{X} = AX + BU \quad (5)$$

For the controlled system described the quadratic performance index is given by:

$$J = \frac{1}{2}\int_0^{\infty}[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)]dt \quad (6)$$

Where Q is appositive semi definite matrix, R is positive definite matrix. For our work we have incorporated degree of stability α. All closed-loop poles are to the left of $-\alpha$. So we define a new performance index cost function

$$J = \frac{1}{2}\int_0^{\infty}[e^{2\alpha T}\{x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\}]dt \quad (7)$$

Where R = $R^T > 0, Q = Q^T \geq 0, M = M^T \geq 0$ are weight matrixes.So defining a new state variable and control

$$u^*(t) = -Kx(t); \quad (8)$$

$$K = -R^{-1}(t)B^T(t)P(t) \quad (9)$$

Where, K is Kalman gain,

The LQR design problem is to design a state feedback controller K such that the objective function *J* (equation 7) is minimized. In this method a feedback gain matrix is designed which minimizes the objective function in order to achieve some compromise between the use of control effort, the magnitude ,and the speed of response that will guarantee a stable system

Using the LQR method, the effect of optimal control depends on the selection of weighting matrices Q and R, if Q and R selected not properly, it make the solution cannot meet the actual system performance requirements. In general, Q and Rare taken the diagonal matrix, the current approach for selecting weighting matrices Q and R is simulation of trial, after finding a suitable Q and R, it allows the use of computers to find the optimal gain matrix K easily.

First we have chosen Q matrix as

Q=$C * C^T$= diag ([1 0 1 0 1 0 1 0]), R=1

Degree of stability α = 0.1

K = [-0.7392 -78.3976 192.3977 -145.2670 -3.8914 -1.0185 7.8722 -20.8247]

Again replacing element of matrix (Q) by

Q = diag ([1400 1200 400 200 0 0 0 0]), R = 1,

Degree of stability α=0.1

[*K, P, E*] = *lqr*(*A, B,Q, R*)

K= [-42.1825 -93.3181 355.7447 -566.4401 -51.9743 -17.6816 -9.3716 -95.9650]

S. No.	Eigen Values of stable system
1	-18.3087+10.9661i
2	-18.3087-10.9661i
3	-13.3256+4.7427i
4	-13.3256-4.7427i
5	-5.3860+1.4948i
6	-5.3860-1.4948i
7	-2.3955+1.2390i
8	-2.3955-1.2390i

Table 2: Eigen values of closed loop system

From table 2 it is clear that the closed loop system is stable since all poles have negative real parts i.e. all poles lie in left half of the s plane. After using LQR controller i.e. Adding gain K and matrix B the system is stable in nature.

V. OBSERVER BASED DESIGN

The response achieved above is good, but was based on the assumption of full-state feedback, which is not necessarily valid. To address the situation where not all state variables are measured, a state estimator must be designed. Before we design our estimator, our system should be observable. Since we know that we can estimate our system state, we will now describe the process for designing a state estimator. The dynamics of the state estimate are described by the following equation.

$$\dot{x} = A\hat{x} + Bu + L(y - \hat{y})$$

The last term corrects the state estimate based on the difference between the actual output y and the estimated output $\hat{y}$. The dynamics of the error in the state estimate is given by

$$\dot{e} = (A - LC)e$$

The error will approach zero if the matrix [A-LC] is stable is with the case for control, the speed of convergence depends on the poles of the estimator Therefore the observer poles should be faster than the controller poles. Commonly estimator poles should be 4 to10 times faster than the slowest controller pole. Based on this logic, we must first find the controller poles. The slowest poles have real part equal to -2.3949 therefore, we will place our estimator poles at -20. Now we will combine our state-feedback controller from before with our state estimator to get the full compensator. The resulting closed-loop system is described by the following matrix equations

$$\begin{bmatrix}\dot{x}\\ \dot{e}\end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}\begin{bmatrix}x\\ e\end{bmatrix} + \begin{bmatrix}B\hat{N}\\ 0\end{bmatrix}r$$

$$y = \begin{bmatrix}C & 0\end{bmatrix}\begin{bmatrix}x\\ e\end{bmatrix} + [0]r$$

P =[-20 -21 -22 -23 -24 -25 -26 -27]

VI. SIMULATION RESULTS

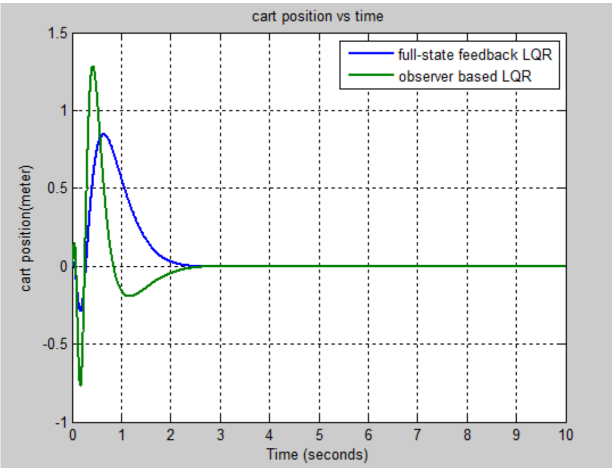


Fig. 3(a): Step response of cart position

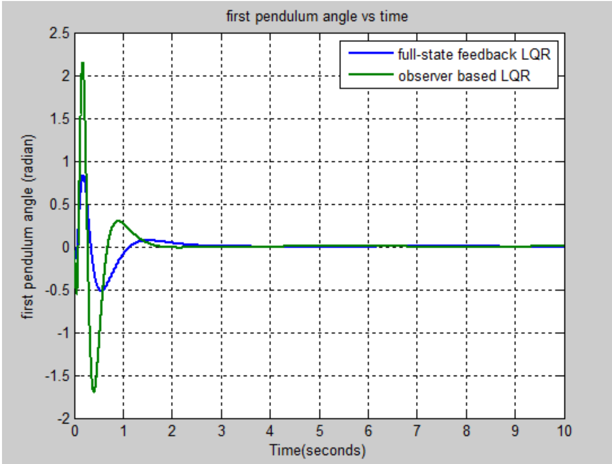


Fig. 3(b): Step response of 1st angle

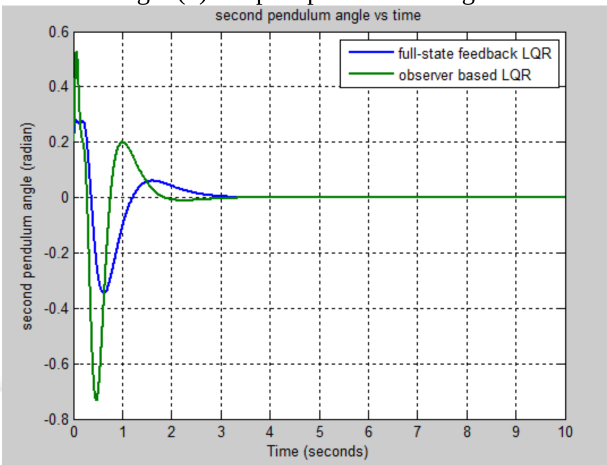


Fig. 3(c): Step response of 2nd angle

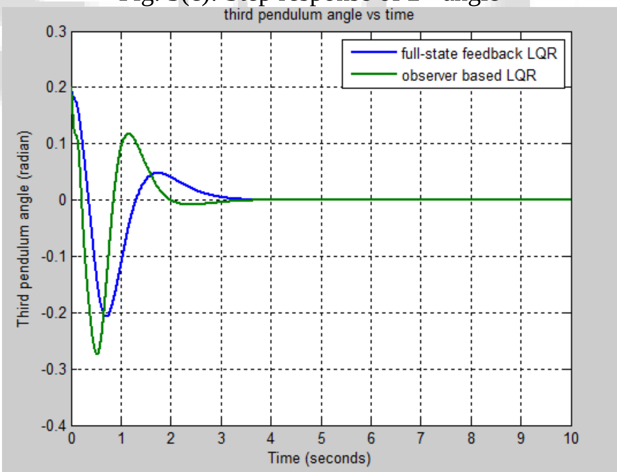


Fig. 3(d): Step response of 3rd angle

The step response of the closed loop system is shown in fig 3(a), 3(b), 3(c), 3(d)

Parameters of TLIP System	LQR	Observer based LQR
Cart Position	2.0892	2.1818
1st pendulum angle	2.4597	1.4890
2nd pendulum angle	2.8133	1.6876
3rd pendulum angle	3.0724	2.9148

Table 3: Settling time (in seconds) for different parameters using LQR and observer based LQR

From the above table it can be clearly seen that the settling time for observer based LQR controller is less than LQR controller. There is a improvement of 39.46%, 40.01%, 5.12% in the settling time of 1st,2nd, 3rd pendulum angle respectively. Angle is of main interest than cart position

Parameters of TLIP System	LQR	Observer based LQR
Cart Position	1.1587e-12	6.2827e-10
1st pendulum angle	0.0050	0.0013
2nd pendulum angle	0.0635	0.0593
3rd pendulum angle	0.2617	0.1774

Table 4: Rise time (in seconds) for different parameters using LQR and observer based LQR

From the above table it can be clearly seen that the rise time for observer based LQR controller is less than LQR controller. There is a improvement of 74%, 6.61%, 32.21% in the settling time of 1st, 2nd, 3rd pendulum angle respectively.

VII. CONCLUSION

In this paper a number of control approaches such as LQR and observer based LQR for CTIPS. The design methods have been successful in meeting the stabilization goals of the CTIPS. From the table it is clear that the rise time and settling time are better for observer based LQR controller as the settling time and rise time obtained are comparatively small when compared to LQR controller. Simulation results clearly establish the effectiveness of the proposed controller as the system performance and stability are satisfactory. The performance of the proposed LQR controller is found to be good and settling time is also small. In this paper the techniques to reduce the settling time and rise time of the system is also discussed. Simulation results clearly show the effectiveness of the proposed controller

VIII. APPENDIX

m_0= 2.4 Kg
m_1=1.323 Kg m_2=1.389 Kg m_3= 0.8655 Kg
L_1=0.402 m, L_2=0.332 m L_3=0.72 m
l_1=0.2449 m l_2=0.193 m l_3=0.3405 m
J_1=.0119 Kgm^2 J_2=.0069Kgm^2 J_3=.0291 Kgm^2
f_0=13.611 Nsm f_1=.0045 Nsm f_2=0.0045 Nsm
f_3=0.0045 Nsm K_s= 9.722 NV $g=9.81ms^{-2}$ [8]

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