On Non – Homogeneous Bi-Quadratic Diophantine Equation $8(x^2+y^2)-15xy=40z^4$

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Abstract— Five different methods of the non-zero integral solutions of the homogeneous biquadratic Diophantine equation with five unknowns $8(x^2 + y^2) - 15xy = 40z^4$ are determined. Introducing the linear transformations x = u + v, y = u – v, u ≠ v ≠ 0 in $8(x^2 + y^2) -15xy = 40z^4$, it reduces to $u^2 +27v^2 = 31z^4$. We are solved the above equation through various choices and are obtained the different methods of solutions which are satisfied it. Some interesting relations among the special numbers and the solutions are exposed.

Key words: Quadratic, Non-Homogenous, Integer Solutions, Special Numbers, Polygonal, Pyramidal Numbers, 2010 Mathematics Subject Classification: 11D09

NOTATIONS USED

$T_{m,n}$: Polygonal number of rank n with sides m.

p_n^5 : Pentagonal pyramidal number of rank n

p_n^6 : Hexagonal pyramidal number of rank n

G_n: Gnomonic number of rank n

$f_{4,3}^r$: Fourth dimensional figurate number of rank n, whose generating Polygon is a Triangle

$f_{4,4}^r$: Fourth dimensional figurate number of rank n, whose generating Polygon is a Square

$f_{4,5}^r$: Fourth dimensional figurate number of rank n, whose generating Polygon is a Pentagon

$f_{4,6}^r$: Fourth dimensional figurate number of rank n, whose generating Polygon is a Hexagon

$f_{4,7}^r$: Fourth dimensional figurate number of rank n, whose generating Polygon is a Heptagon

$f_{4,8}^r$: Fourth dimensional figurate number of rank n, whose generating Polygon is a Octagon

I. INTRODUCTION

The number theory is the queen of mathematics. In particular, the Diophantine equations have a blend of attracted interesting problems. For an extensive review of variety of problems one may refer to [3-12].In 2014.In 2014 Jayakumar. P Sangeetha. K [12] have published a paper in finding the integer solutions of the homogeneous Bi-quadratic Diophantine equation $(x^3 - y^3)z = (w^2 - p^2)R^4$. In 2015 Jayakumar P, Meena J [14,15] published two papers in finding the integer solutions of the solutions of the homogeneous Bi-quadratic Diophantine equation $(x^4 - y^4)z = 26(z^2 - w^2)R^2$ and $(x^4 - y^4)z = 40(z^2 - w^2)R^2$ Inspired by these, in this work, we are observed another interesting solutions five different methods of the non- zero integral solutions of the non-homogeneous bi-quadratic Diophantine equation with three unknowns $8(x^2 + y^2) - 15xy = 40z^4$. Further, some elegant properties among the special numbers and the solutions are exposed.

II. DESCRIPTION OF METHOD

Consider the bi-quadratic Diophantine equation

$$8(x^2 + y^2) - 15xy = 40z^4 \qquad (1)$$

Introduce the linear transformations

$$x = u + v, \quad y = u - v \qquad (2)$$

Using (2) in (1), this gives to

$$u^2 + 31v^2 = 40z^4 \qquad (3)$$

We solved (3) through various choices and the different methods of solutions of (1) are obtained as follows.

A. *Method 1:*

Consider (3) as $u^2 + 31v^2 = 31z^4 + 9z^4$ and take it as in the form of ratio as

$$\frac{u+3z^2}{31(z^2+v)} = \frac{z^2-v}{(u-3z^2)} = \frac{a}{b}, \ b \neq 0 \qquad (4)$$

The equation (4) is equivalent to the system of equation as

$$bu - 31av + (3b - 31a) z^2 = 0 \qquad (5)$$

$$-au - b v + (3a + b) z^2 = 0 \qquad (6)$$

Solving (5) and (6) then applying cross rule method, we have

$$u = 93a^2 - 3b^2 + 62ab$$

$$v = -31a^2 + b^2 + 6ab$$

$$z^2 = 31a^2 + b^2 \qquad (7)$$

If we substitute, $a = 2pq$, $b = 31p^2 - q^2$ in (7) and using (2), then we get

- $x = x(p, q) = -1922 p^4 - 2q^4 + 372p^2q^2 + 4216p^3q - 136p$
- $y = y(p, q) = -3844p^4 - 4q^4 + 744p^2q^2 + 3472p^3q - 112pq^3$ $z = z(p, q) = 31 p^2 + q^2$,

Which gives us the non- zero distinct integer solutions to (1)

1) *Observations*

- $x(p,1) + 46128 f_{4,7}^p - 7688 T_{4,p^2} - 62248 p_p^5 + 17298T_{4,p} + G_{1990p} + 3 = 0$.
- $x(1, p) + 48 f_{4,5}^p - 4 T_{4,p^2} + 232 p_p^5 - 506T_{4,p} - G_{2110p} \equiv 17 (\text{Mod } 113)$
- $y(1, p) + 96 f_{4,8}^p - 20 T_{4,p^2} + 96 p_p^5 - 816T_{4,p} - G_{1728p} =$ cubic integer.
- $y(p,1) + 46128 f_{4,4}^p - 37696 p_p^5 - 1116T_{4,p} - G_{3788p} + 3 = 0$.
- $x(1, p) + y(1, p) + 144 f_{4,3}^p + 424 p_p^5 - 1394T_{4,p} - G_{3862p} \equiv 0 (\text{Mod } 5)$
- $x(p,1) - y(p,1) - 11532 f_{4.6}^p + 10044 p_p^5 - 806T_{4,p} + G_{12p} = 1$.
- $\frac{9}{8} z(2,2)$ is a perfect square.

- $z(3,3)$ is a cubic integer.
- $z(2,6)-1$ is a Woodall number.
- *$z(1,1)-2$ is a Nasty number.*

B. Method 2:

Instead of (4) we take the form of ratio as

$$\frac{u+3z^2}{z^2-v}=\frac{31(z^2+v)}{u-3z^2}=\frac{a}{b},\ b\neq 0 \qquad (8)$$

The following procedure similar to method I, the corresponding integer solutions to (1) are found as

- $x = x(p,q) = -119164p^4 - 124q^4 + 7704p^2q^2 + 3472p^3q - 112pq^3$
- $y = y(p,q) = -59582p^4 - 62q^4 + 3852p^2q^2 + 4216p^3q - 136pq^3$
- $z = z(p,q) = 31p^2 + q^2$

1) Observations

- $x(p,1)+2859936 f^p_{4,8} - 595820 T_{4,p^2} - 3820192 p^5_p + 1187408T_{4,p} + G_{238384p} \equiv 0 \pmod 5$.
- $x(1,p) + 2976 f^p_{4,3} - 1264 p^5_p - 8436T_{4,p} - G_{2108p} \equiv (\mathrm{Mod}\ 39721)$
- $y(1,p)+744 f^p_{4,4} - 224 p^5_p - 4050 T_{4,p} - G_{2170p} \equiv 1 \pmod 5$
- $x(1,p) - y(1,p) + 372 f^p_{4,6} - 420 p^5_p - 3766T_{4,p} + G_{372p} \equiv 3 \pmod{19861}$
- $y(p,1)+1429968 f^p_{4,7} - 238328 T_{4,p^2} - 1676728 p^5_p + 417438T_{4,p} + G_{59650p}$ = woodall number
- $x(p,1)+y(p,1)+1072476 f^p_{4,6} - 1087852 p^5_p + 174878T_{4,p} + G_{124p} \equiv 11 \pmod{17}$
- $\frac{3}{4}z(1,1)$ is a Nasty number
- $z(p,2) - 31T_{4,p}$ is a perfect square.
- $x(1,p) - 2y(1,p) + G_{248p}$ = Cullen number
- $\frac{1}{125}z(10,5)$ is a perfect square.

C. Method 3:

Let us assume

$$40 = (2 + i\sqrt{27})(2 - i\sqrt{27}) \qquad (9)$$

Take z as

$$z = z(a,b) = a^2 + 31b^2 \qquad (10)$$

Using (9) and (10) is (3) then applying the method of factorization, define

$$(u + i\sqrt{31}v) = (3+i\sqrt{31})(a + i\sqrt{31}b)^4$$

This gives us

$$u = 3a^4 + 2883b^4 - 558a^2b^2 - 124a^3b + 3884ab^3$$
$$v = a^4 + 961b^4 - 186a^2b^2 + 12a^3b - 372ab^3 \qquad (11)$$

Using (11) in (2), the corresponding integer solutions to (1) are found as

- $x = x(a,b) = 4a^4 + 3884b^4 - 744a^2b^2 - 112a^3b + 3472a^3$
- $y = y(a,b) = 2a^4 + 1922b^4 - 372a^2b^2 - 136a^3b + 4216ab^3$
- $z = z(a,b) = a^2 + 31b^2$

1) Observations:

- $x(A,1) - 96 f^A_{4,3} + 272 p^5_A + 652T_{4,A} - G_{1724A} \equiv 0 \pmod 5$
- $y(A,1) - 48 f^A_{4,8} + 10T_{4,A^2} + 336 p^5_A + 216T_{4,A} - G_{2112A} \equiv 0 \pmod 3$
- $x(A,1) - y(A,1) - 48 f^A_{4,7} + 8T_{4,A^2} + 8 p^5_A + 382T_{4,A} + G_{370A} \equiv 3 \pmod{641}$
- $x(1,A) - 23064 f^A_{4,6} + 16120 p^5_A + 372T_{4,A} + G_{56A} + 3 = 0$
- $y(1,A) - 38444 f^A_{4,4} + 69444 p^5_A + 6510T_{4,A} + G_{1990A} = 3$
- $x(1,A) - y(1,A) - 11532 f^A_{4,6} + 13020 p^5_A - 2294T_{4,A} - G_{12A} = 3$.
- $z(A+1,A) - 32T_{4,A} - G_{2A}$ is a jacobsthal Lucas number
- $8z(1,1)$ is a perfect square
- $9z(0,1)$ is a cubic integer
- $\frac{1}{12}z(3,3)$ is a nasty number

D. Method 4:

In place of (9) take 31 as

$$40 = \frac{(63+i\sqrt{31})(63-i\sqrt{31})}{100} \qquad (12)$$

The Following procedure is similar to pattern III, the corresponding integer solutions to (1) are found as

$$u = \frac{1}{10}[63a^4+60543b^4-11718a^2b^2-124a^3b+3844ab^3] \qquad (13)$$

$$v = \frac{1}{10}[a^4+961b^4-186a^2b^2+252a^3b-7812ab^3] \qquad (14)$$

In true of (2), the values x and y are

$$x = \frac{1}{10}[64a^4+615043b^4-11904a^2b^2+128a^3b-3968ab^3] \qquad (15)$$

$$y = \frac{1}{10}[62a^4+59582b^4-11532a^2b^2-376a^3b+11656ab^3] \qquad (16)$$

Since our intension is to find integer solutions, taking 'a' as 10A & 'b' as 10B in(15) & (16), the relating parametric integer values of (1) are found as

- $x=x(A,B)=64000A^4+61504000B^4-11904000A^2B^2+128000A^3B-3968000AB^3$
- $y = y(A,B) = 62000A^4+59582000B^4-11532000A^2B^2-376000A^3B+11656000AB^3$
- $z = z(A,B) = 196A^2 + 6076B^2$

1) Observations

- $x(A,1) - 1536000 f^A_{4,3} + 512000 p^5_A + 12352000T_{4,A} + G_{2176000A} \equiv 3 \pmod{20501333}$
- $y(A,1) - 744000 f^A_{4,4} + 936000 p^6_A + 11374000T_{4,A} - G_{5688000A} \equiv 3 \pmod{19860667}$
- $x(A,1) - y(A,1) - 48000 f^A_{4,5} + 4000T_{4,A^2} - 968000 p^5_A + 874000T_{4,A} + G_{7814000A} \equiv 1 \pmod 3$
- $x(1,A) - 1476096000 f^A_{4,8} + 307520000T_{4,A^2} + 1976064000 p^5_A - 607104000T_{4,A} - G_{123072000A} \equiv 7 \pmod{9143}$.
- $y(1,A) - 1429968000 f^A_{4,7} + 238328000T_{4,A^2} + 1644984000 p^5_A - 393886000T_{4,A} - G_{59394000A}$, is a cubic integer.
- $x(1,A) - Y(1,A) - 11532000 f^A_{4,6} + 42780000 p^5_A - 17174000T_{4,A} - G_{252000A} \equiv 3 \pmod{667}$
- $z(1,0)$ is a perfect square.
- $\frac{1}{196}Z(1,1) - 2$ is a nasty number.
- $z[A(A+1),1] - 196(p_A)^2 \equiv 0 \pmod 2$
- $z(A+1,A)+6264T_{4,A} + ct_{16,A} + G_{192A}$ is a perfect square.

E. Method 5:

Let us Consider (3) as

$$u^2 + 31v^2 = 40z^4 * 1 \quad (17)$$

Take 1 as $1 = \frac{(9+i7\sqrt{31})(9-i7\sqrt{31})}{1600}$ (18)

Using (9), (10) and (14) in (13) then applying the method of factorization process, we define

$$(u + i\sqrt{27}v) = (2+ i\sqrt{27})\,(a + i\sqrt{27}b)^4\left(\frac{13+i\sqrt{27}}{14}\right).$$

It furnishes us

$$u = \frac{1}{40}\,[-190a^4 -182590b^4 +35340a^2b^2 +115320ab^3 - 3720a^3b] \quad (19)$$

$$v = \frac{1}{40}[30a^4+28830b-5580a^2b^2+23560ab^3-760a^3b] \quad (20)$$

In sight of (2), the values of x & y as

$$x = \frac{1}{40}[-160a^4-153760b^4+29760a^2b^2 + 138880ab^3 - 4480a^3b]. \quad (21)$$

$$y = \frac{1}{40}\,[-220a^4 -211420b^4 +40920a^2b^2 + 91760ab^3 - 2960a^3b]. \quad (22)$$

As our intension is to find integer solutions, taking ‘a’ as 40A & ‘b’ as 40B in (21) & (22), the relating parametric integer values of (1) are found as

- $x=x(A,B)=-160A^4-153760B^4+ 29760A^2B^2+138880AB^3-4480A^3B$
- $y=y(A,B)=-220A^4 - 211420B^4 +40920A^2B^2+91760AB^3 -2960A^3B$
- $z = z(A, B) = 100A^2 + 3100B^2$

1) Observations

- $x(A, 1) +3840 f^{A}_{4,8} -4800 f^{A}_{4,6} + 8640 p^{5}_{A} + 28000T_{4,A} - G_{69120A} \equiv 3 \pmod{51253}$
- $y(A,1)+5280 f^{A}_{4,7} -880 T_{4,A^2} -240 p^{5}_{A} -42340T_{4,A}-G_{45660A}$, is a cubic integer.
- $x(A,1)-Y(A,1)-1440 f^{A}_{4,5} +120 T_{4,A^2} +4240 p^{5}_{A} +9580T_{4,A} - G_{23500A} \equiv 23 \pmod{2507}$
- $x(1,A)+3690240 f^{A}_{4,3} -2122880 p^{5}_{A} -659680T_{4,A}-G_{459040A}$, is a wood all number.
- $y(1,A)+2537040 f^{A}_{4,4} -1874880 p^{5}_{A} -160580T_{4,A}+G_{212900A} \equiv 13 \pmod{17}$
- $x(1,A)-Y(1,A)-345960 f^{A}_{4,6} +440200 p^{5}_{A} - 93620T_{4,A} - G_{760A} \equiv 1 \pmod{2}$
- $z(1,0)$ is a perfect square.
- $\frac{1}{10} z(1,1) + 10$ *is a Nasty number*.
- $\frac{1}{50} z(2,,2)$ is a Perfect square.
- $z(A,A + 1)-3200T_{4,A} - G_{3100A} \equiv 7 \pmod{443}$

III. Conclusion

In this paper, we have observed various process of determining infinitely a lot of non-zero different integer values to the non-homogeneous bi-quadratic Diophantine equation $8(x^2+y^2)-15xy = 40z^4$.One may try to find non-negative integer solutions of the above equations together with their similar observations.

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