On Statistically Convergent Sequences of Real Numbers

Dr C Vijender
Assistant Professor
Department of Mathematics
Sreenidhi Institute of Science and Technology, Hyderabad-501301

Abstract— In this present paper on statistically convergent sequences of Real numbers, convergent of a sequence and Cauchy's convergent theorem and convergent of monotonic sequences and generalization of Cauchy's criterion.

Key words: Sequence, monotonic sequence, Cauchy's criterion

I. INTRODUCTION

Definition: A *sequence* of real numbers is any function $a : \mathbf{N} \to \mathbf{R}$. However, we usually write a_n for the image of n under a, rather than $a(n)$. The values a_n are often called the *elements* of the sequence. To make a distinction between a sequence and one of its values it is often useful to denote the entire sequence by $(a_n)_{n=1}^{\infty}$, or just (a_n). When specifying a particular sequence, it may be written in the form $(a_1, a_2, a_3, \ldots)$, where sufficiently many elements of the sequence are given so that the pattern is clear.

For example, (1, 2, 3, 4, …), (1, -2, 3, -4, …), and $(1, \pi, \pi^2, \pi^3, \pi^4, \ldots)$ are all sequences. Note, however, that there need not be any particular pattern to the elements of the sequence. For example, we may specify a_n to be the n-th digit of π. Often it is useful to specify a sequence recursively. That is, to specify some initial values of the sequence, and then to specify how to get the next element of the sequence from the previous elements. For example, consider the sequence $x_1=1$, $x_2=1$, and $x_n = x_{n-1} + x_{n-2}$ for $n \geq 3$. This sequences is known as the Fibonacci sequence, and its first few terms are given by (1, 1, 2, 3, 5, 8, 13, …). We can also perform algebraic operations on sequences. In other words, we can add, subtract, multiply, divide sequences. These operations are simply performed element by element, for completeness we give the definitions.

II. PRELIMINARY NOTES

Definition Given two sequences (x_n) and (y_n) and a real number c, we define the following operations:

- $(x_n)+(y_n)=(x_n+y_n)$;
- $(x_n)-(y_n)=(x_n-y_n)$;
- $(x_n)\cdot(y_n)=(x_n\cdot y_n)$;
- if $y_n \neq 0$ for all n in $\mathbf{N}$, $(x_n)/(y_n)=(x_n/y_n)$;
- $c\cdot(x_n)=(c\cdot x_n)$

A. Classification of Sequences

Some properties of sequence are so important that they are given special names. A sequence (a_n) is called:

- strictly increasing if $a_n < a_{n+1}$ for all n in **N**;
- non-decreasing if $a_n \leq a_{n+1}$ for all n in **N**;
- strictly decreasing if $a_n > a_{n+1}$ for all n in **N**;
- non-increasing if $a_n \geq a_{n+1}$ for all n in **N**;
- monotone if it satisfies any of the above properties, that is, if it is either non-decreasing or non-increasing;
- strictly monotone if it is either strictly increasing or strictly decreasing;
- bounded above if there exists M in **R** such that $a_n<M$ for all n in **N**.
- bounded below if there exists M in **R** such that $a_n>M$ for all n in **N**.
- bounded if it is both bounded above and bounded below.
- Cauchy if for all $\varepsilon>0$ there exists a natural number N so that, for all n, m > N, $|a_m-a_n| < \varepsilon$.

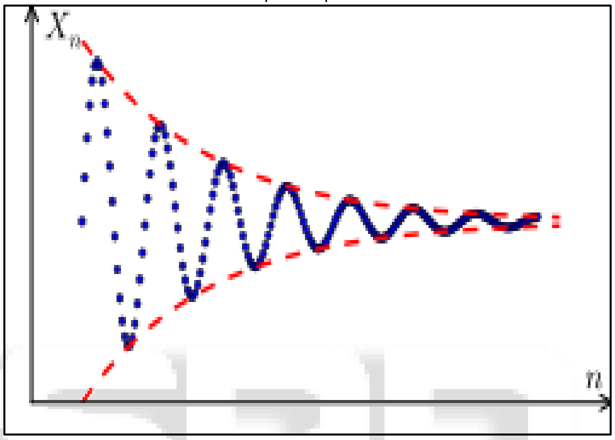

Fig. 1:

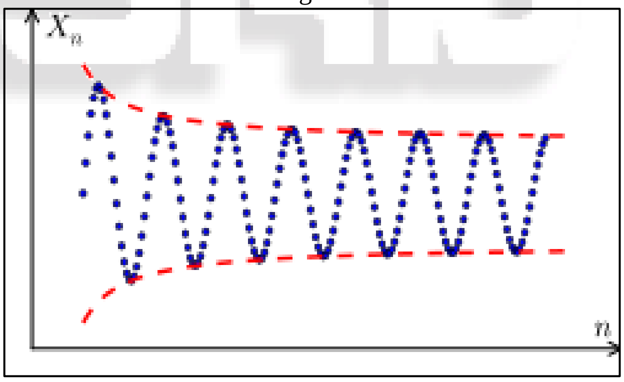

Fig. 2:

B. Convergence

Definition Let (x_n) be a sequence of real numbers. The sequence (x_n) is said to *converge* to a real number a.
if for all $\varepsilon>0$, there exists N in $\mathbf{N}$ such that $|x_n-a|<\varepsilon$ for all $n \geq N$.
If (x_n) converges to a then we say a is the *limit* of (x_n) and write

$$\lim_{n\to\infty} x_n = a$$

or

$x_n \to a$ as $n \to \infty$.

This is read x_n *approaches a* as n *approaches* ∞. If it is clear which variable is playing the role of n then this may be abbreviated to simply $x_n \to a$ or $\lim x_n=a$.
If a sequence converges, then it is called *convergent*.

It is also useful to extend this concept and allow sequences whose limits are either ∞ or −∞

Definition We say $x_n \to \infty$ as $n \to \infty$ if for every M in **R** there is a natural number N so that $x_n \geq M$ for all $n \geq N$. We say $x_n \to -\infty$ as $n \to \infty$ if for every M in **R** there is a natural number N so that $x_n \leq M$ for all $n \geq N$.

Despite this, we do not refer to sequences such as these as convergent.

The following theorems tells us that a convergent sequence converges to exactly one number. This may seem intuitively clear, mathematically we have to prove that limits behave the way we expect. After all, we gave the definition and nothing tells us ahead of time that it is correct.

III. MAIN RESULT

A. Theorem (Uniqueness of limits)

A sequence can have at most one limit. In other words: if $x_n \to a$ and $x_n \to b$ then $a = b$.

Proof:

Suppose the sequence has two distinct limits, so $a \neq b$. Let $\varepsilon = |a-b|/3$.

Certainly $\varepsilon > 0$, using the definition of convergence twice we can find natural numbers N_a and N_b so that

$$|x_n - a| \leq \epsilon \text{ for all } n > N_a.$$

and

$$|x_n - b| \leq \epsilon \text{ for all } n > N_b.$$

Taking $k = \max(N_a, N_b)$ then both of these conditions hold for x_k. Hence we deduce that $|x_k - a| \leq \varepsilon$ and $|x_k - b| \leq \varepsilon$. Applying the triangle inequality, we see

$$|a-b| = |(x_k - b) - (x_k - a)| \leq 2\epsilon = (2/3)|a-b|,$$

which is a contradiction. Thus, any sequence has at most one limit. □

Theorem (Convergent Sequences Bounded)

If $(x_n)_{n=1}^{\infty}$ is a convergent sequence, then it is bounded.

Proof

Let $a = \lim_{n\to\infty} x_n$, and let $\varepsilon = 1$.

From the definition of convergence there exists a natural number N such that

$$|x_n - a| \leq 1 \text{ for all } n \geq N.$$

The sequence $(x_n)_{n=N+1}^{\infty}$ is bounded above by $a+1$ and below by $a-1$. Let $M = \max(|x_1|, |x_2|, |x_3|, \ldots, |x_N|, |a|+1)$. It follows that $-M \leq x_n \leq M$ for all n in **N**. Hence the sequence is bounded.

B. Theorem (Boundedness of Cauchy Sequences)

If $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence, then it is bounded.

Proof

Let (x_n) be a Cauchy sequence. By the definition of a Cauchy sequence, there is a natural number N such that $|x_n - x_m| < 1$ for all $n, m > N$. In particular, $|x_{N+1} - x_m| < 1$

for all $m > N$. It follows by the reverse triangle inequality that $|x_m| < |x_{N+1}| + 1$. If we take $M = \max(|x_1|, |x_2|, \ldots, |x_N|, |x_{N+1}| + 1)$, then $|x_n| \leq M$ for all n in **N**.

C. Theorem (Algebraic Operations)

If (x_n) and (y_n) are convergent sequences and $c \in \mathbf{R}$, the following properties hold:

$$\lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} (x_n) + \lim_{n\to\infty} (y_n).$$

$$\lim_{n\to\infty} (x_n y_n) = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} y_n\right).$$

$$\lim_{n\to\infty} (a x_n) = a \lim_{n\to\infty} (x_n).$$

$$\lim_{n\to\infty} \left(\frac{x_n}{y_n}\right) = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$$

(assuming $y_n \neq 0$ for all n in **N** and $\lim y_n \neq 0$).

If $x_n \leq y_n$ for every n in N, then $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$.

1) Proof

Let $x = \lim x_n$ and $y = \lim y_n$. We need to show that for any $\varepsilon > 0$ there is natural number N so that if $n \geq N$, then $|(x_n + y_n) - (x + y)| \leq \varepsilon$. Given any $\varepsilon > 0$ we have $\varepsilon/3 > 0$ so from the definition of convergence there is a natural number N_x so that $|x_n - x| \leq \varepsilon/3$ for all $n > N_x$, similarly we can choose N_y $|y_n - y| \leq \varepsilon/3$ for all $n > N_y$.

Let $N = \max(N_x, N_y)$. If $n > N$, then by the triangle inequality we have

$$|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)| \leq \epsilon/3 + \epsilon/3 < \epsilon,$$

which is what we needed to show.

Let $x = \lim x_n$ and $y = \lim y_n$. Since these sequences are convergent they are bounded. Let M_x be a bound for (x_n) and let M_y be a bound for (y_n). By increasing these quantities of necessary we may also assume $M_x > x$ and $M_y > y$. Given $\varepsilon > 0$, there exists some N_x and N_y such that

$$|x_n - x| < \frac{\epsilon}{2M_y} \text{ for } n > N_x \text{ and}$$

$$|y_n - y| < \frac{\epsilon}{2M_x} \text{ for } n > N_y.$$

Then for every $n > \max(N_x, N_y)$,

$$\begin{aligned} |x_n y_n - xy| &= |(x_n - x)y_n + x(y_n - y)| \\ &\leq |x_n - x| M_y + M_x |y_n - y| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon. \end{aligned}$$

Let $y_n = a$ for all n in **N**. The statement now follows from 2.

We can reduce this to showing that $\lim (1/y_n)$ exists and equals $1/(\lim y_n)$. Then it follows by 2 that we have:

$$\lim_{n\to\infty} \left(\frac{x_n}{y_n}\right) = \left(\lim_{n\to\infty} \frac{1}{y_n}\right)\left(\lim_{n\to\infty} x_n\right) = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}.$$

Let $y = \lim y_n$. By the exercises, since y and y_n are not 0, we can find $\delta > 0$ so that $|y\_n| > \delta$ and $|y| > \delta$. It follows that $1/|y_n y| < 1/\delta^2$. Given $\varepsilon > 0$ choose n in **N** so that $|y_n - y| < \delta^2 \varepsilon$. We have

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| = |y - y_n| \frac{1}{|y_n y|} \leq \frac{|y - y_n|}{\delta^2} < \epsilon$$

Hence, $\lim_{n\to\infty} \frac{1}{y_n} = \frac{1}{y}$.

We first can reduce to the case when one sequence is identically 0. To see this let $z_n = x_n - y_n$. Then $z_n < 0$ for all n in $\mathbf{N}$. Let $z = \lim z_n$. Suppose that $z > 0$ then we can then find an natural number N so that

$$|z - z_N| < z.$$

Since $z_N \leq 0 < z$, the absolute value equals $z - z_N$. Subtracting z we find that $-z_N < 0$. Hence z_N is positive. Contradiction. Therefore we must have that $z \leq 0$. Which means that by 1 we get:
Therefore $\lim x_n \leq \lim y_n$

D. Theorem (Squeeze/Sandwich Limit Theorem)

Given sequences (x_n), (y_n), and (w_n), if (x_n) and (y_n) converge to a and $x_n \leq w_n \leq y_n$, then w_n converges to a.

1) Proof

Fix $\varepsilon > 0$. We need to find an N such that $|w_n - a| < \varepsilon$ if $n > N$. Since $(x_n) \to a$ and $(y_n) \to a$ the definition of convergence ensures that there exists integers N_x and N_y so that $|x_n - a| < \varepsilon$ for $n > N_x$ and $|y_n - a| < \varepsilon$ for $n > N_y$.

Let N=max(N_x, N_y). Then, forall $n > N$ we have $\varepsilon < x_n - a$ and $y_n - a < \varepsilon$. Since $x_n < w_n < y_n$, it follows that $x_n - a < w_n - a < y_n - a$.

Thus if $n \geq N$, then $-\varepsilon < x_n - a < w_n - a < y_n - a < \varepsilon$. In other words, $|w_n - a| < \varepsilon$.

2) Completeness

The following results are closely related to the completeness of the real numbers.

E. Theorem (Convergence of Monotone sequences)

Any monotone, bounded sequence converges. If the sequence is non-decreasing, then the sequence converges to the least upper bound of the elements of the sequence. If the sequence is non-increasing, then the sequence converges to the greatest lower bound of the elements of the sequence

1) Proof

Let (x_n) be any monotone sequence that is bounded by a real number M. Without loss of generality, assume (x_n) is non-decreasing. Since (x_n) is bounded above, it has a least upper bound by the least upper bound axiom. Let $x = \sup\{x_n \mid n \in \mathbf{N}\}$. We will now show that $(x_n) \to x$.
Fix $\varepsilon > 0$. As was shown in the exercises, if $s = \sup(A)$, then for any $\varepsilon > 0$ there is an element a in A so that $s - \varepsilon < a < s$. Hence, it follows that there exists an N in $\mathbf{N}$ so that $x - \varepsilon < x_N < x$.
For any $n > N$, since x_n is non-decreasing, we have that

$$x - \epsilon < x_N \leq x_n < x.$$

Thus $|x - x_n| < \varepsilon$ and by the definition of convergence, (x_n) converges to x.

F. Theorem (Nested intervals property)

If there exists a sequence of closed intervals $I_n = [a_n, b_n] = \{x \mid a_n \leq x \leq b_n\}$ such that $I_{n+1} \subseteq I_n$ for all n, then $\cap I_n$ is nonempty.

1) Proof

Since $I_{n+1} \subseteq I_n$ it follows that $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$.
Since (a_n) and (b_n) are monotonic sequences they converge by the previous theorem. Furthermore, since $a_n < b_n$ for all n, it follows that $\lim a_n \leq \lim b_n$.
By the monotonicity of (a_n) and (b_n) we have for every n

$$a_n \leq \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n \leq b_n.$$

Therefore $\lim a_n \in [a_n, b_n]$ for every n, which implies that

$$\lim_{n\to\infty} a_n \in \bigcap_{n=1}^{\infty} I_n.$$

Thus the intersection is nonempty.

G. Theorem (Bolzano—Weierstrass)

Every bounded sequence of real numbers contains a convergent subsequence.

1) Proof

Let (x_n) be a sequence of real numbers bounded by a real number M, that is $|x_n| < M$ for all n. We define the set A by $A = \{r \mid |r| \leq M$ and $r < x_n$ for infinitely many $n\}$. We note that A is non-empty since it contains $-M$ and A is bounded above by M. Let $x = \sup A$.

We claim that, for any $\varepsilon > 0$, there must be infinitely many points of x_n in the interval $(x - \varepsilon, x + \varepsilon)$. Suppose not and fix an $\varepsilon > 0$ so that there are only finitely many values of x_n in the interval $(x - \varepsilon, x + \varepsilon)$. Either $x \leq x_n$ for infinitely many n or $x \leq x_n$ for at most only finitely many n (possibly no n at all). Suppose $x < x_n$ for infinitely many n. Clearly in this case $x \neq M$. If necessary restrict ε so that $x + \varepsilon \leq M$. Set $r = x + \varepsilon/2$ we have that $r < x_n$ for infinitely many n because there are only finitely many x_n in the set $[x,r]$ and x must be less than infinitely many x_n, furthermore $|r| < M$. Thus r is in A, which contradicts that x is an upper bound for A. Now suppose $x < x_n$ for at most finitely many n. Set $y = x - \varepsilon/2$. Then there are at most only finitely man n so that $x_n \geq y$. Thus, if $r < x_n$ for infinitely many n, we have that $r \leq y$. This means that y is an upper bound for A that is less than x, contradicting that x was the *least* upper bound of A. In either case we arrive at a contradiction, thus we must have that for any $\varepsilon > 0$, there must be infinitely many points of x_n in the interval $(x - \varepsilon, x + \varepsilon)$.

Now we show there is a subsequence that converges to x. We define the subsequence inductively, choose any x_{n1} from the interval $(x - 1, x + 1)$. Assuming we have chosen $x_{n1}, \ldots, x_{nk-1}$, choose x_{nk} to be an element in the interval $(x - 1/k, x + 1/k)$ so that $n_k \notin \{n_1, \ldots, n_{k-1}\}$, this is possible as there are infinitely many elements of (x_n) in the interval. Notice that for this choice of x_{nk} we have that $|x - x_{nk}| < 1/k$. Hence for any $\varepsilon > 0$, if we take any $k > 1/\varepsilon$, then $|x_{nk} - x| < \varepsilon$. That is the subsequence $(x_{nk}) \to x$.

H. Theorem (Cauchy criterion)

A sequence converges if and only if it is Cauchy. Although this seems like a weaker property than convergence, it is actually equivalent, as the following theorem shows:

1) Proof

First we show that if $(x_n) \to x$ then x_N is Cauchy. Now suppose that for a given $\varepsilon > 0$ we wish to find an N so that $|x_n - x_m| < \varepsilon$ for all $n, m > N$. We will choose N so that for all $n \geq N$ we have that $|x_n - x| < \varepsilon/2$. By the triangle inequality, for any $n, m > N$ we have:
Thus (x_n) is a Cauchy sequence.

$$|x_n - x_m| \leq |x_n - x| + |x_m - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Now we show that if (x_n) is a Cauchy sequence, then it converges to some x. Let (x_n) be a Cauchy sequence, and let ε > 0. By the definition of a Cauchy Sequence, there exits a natural number L so that $|x_n - x_m| < \varepsilon/2$ whenever $n, m > L$. Since (x_n) is a Cauchy sequence it is bounded. By the Bolzano—Weierstrass theorem, it has a convergent to some point x. Now we will show that the whole Because (x_{nk}) converges, we can choose a natural number M so that if $n_k > M$, then $|x_{nk} - x| < \varepsilon/2$. Let $N = \max(L, M)$, and fix any $n_k > N$. For $n > N$ we have that

$$|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus by definition of convergence $(x_n) \to x$.

These theorems all describe different aspects of the completeness of the real numbers. The reader will notice that the least upper bound property was used heavily in this section, and it is the axiom that separates the real numbers from the rational numbers. While these theorems would be false for the rational numbers, not all of them can substitute for the least upper bound property. The Cauchy criterion and the nested intervals property are not strong enough to imply the least upper bound property without additional assumptions, while the Convergence of Monotone sequences theorem and the Bolzano—Weierstrass property do imply the least upper bound property.

It is nice exercise to prove the following equivalence for the real number system:

1) LUB/supremum property
2) Monotone Convergence property
3) Nested Interval property
4) Bolzano Weierstrass property
5) Cauchy Criterion property

Pro0f: (i)⟹(ii)⟹(iii)⟹(iv)⟹(v) ⟹(i)

Given a non-empty set $T \subset \mathbb{R}$ with an upper bound, $y0$. Let $x0$ be any elment of T.

Given xn and yn, define $zn=(xn+yn)/2$.

If zn is an upper bound for T, then let $xn+1=xn$ and $yn+1=zn$.

If zn is not an upper bound for T, let $xn+1>zn$ be a member of T greater than zn, and let $yn+1=yn$.

Lemma: For every n, $|xn-yn| \leq |x0-y0|/2^n$

Lemma: If $m>n$, then $|ym-yn| \leq |x0-y0|/2^n$

So $\{yn\}$ is Cauchy. Now you just have to prove the limit of $\{yn\}$ is the LUB of T. (Hint: By the same reasoning, $\{xn\}$ is Cauchy, and the two limits must be equal.)

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