Common Fixed Point Theorems in Fuzzy Metric Space Satisfying φ-Contractive Condition with JCLR Property

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Abstract— The objective of this, we utilize the joint common limit range property of mapping called (JCLR) Property. some illustrative examples are furnished which demonstrate the validity of utility of our results. As an application to our main result, we prove an some type of fixed point theorem in fuzzy metric space. Our results improve and extend the host of previously known results including the ones contained in Imdad et al., cho.et.al and manro et.al.

Key words: Fixed Point, Fuzzy Metric Space, Weakly Compatible, Coincidence Point, E.A Property, CLR Property, JCLR Property

I. INTRODUCTION

The concept of fuzzy set was received by Zadeh [1] in 1965. After that, it was developed extensively by many researchers, which also include interesting application of this theory in different fields. Fuzzy set theory has application in applied science such as neural network theory, stability theory, mathematical programming, modelling theory, engineering sciences, medical sciences (medical genetics, nervous system), image processing, control theory, communication. In 1975, Kramosil and Michalek [2] studied the notion of fuzzy metric space which could be considered as generalization of probabilistic metric space due to Menger [3]. Fixed point theory in fuzzy metric spaces has been developed starting with the work of Heilpern [4]. In [5,6] George and Veeramani modified the notion given by kramosil and Michalek, in order to introduce a Hausdorff topology on fuzzy metric space. Other Many authors have contributed to the development of this theory and its applications, for instance [7,8,9,10,11,12,13,14,15,16,17,18, 19,20].

In 2002, Amari and El Moutawakil [21] defined the notion of (E.A) property for self mappings which contained the class of non-compatible mappings in metric spaces. It was pointed out that (E.A) property allows replacing the completeness requirement of the space with a more natural condition of closedness of the range as well as relaxes the complexity of the whole space, continuity of one or more mappings and containment of the range of one mapping into the range of other which is utilized to construct the sequence of joint iterates. Subsequently, there are a number of results proved for contraction mappings satisfying (E.A) property in fuzzy metric spaces (see [22-27]). Most recently, Sintunavarat and Kumam [28] defined the notion of “common limit in the range” property (or (CLR) property) in fuzzy metric spaces and improved the results of Mihet [26]. In [28], it is observed that the notion of (CLR) property never requires the condition of the closedness of the subspace while (E.A) property requires this condition for the existence of the fixed point (also see [29]). Many authors have proved common fixed point theorems in fuzzy metric spaces for different contractive conditions.

In this paper, we prove common fixed point theorems for weakly compatible mapping with joint common limit range property in fuzzy metric space. Some related results are also derived besides furnishing illustrative examples. We also present some integral type common fixed point theorems in fuzzy metric space. Our results improve extended and generalize a host of previously known results existing in the literature.

II. PRELIMINARIES

A. Definition 1 [30]
A binary operation * : [0, 1] x [0, 1] → [0, 1] is said to be continuous t-norm if it satisfies the following condition:
- is commutative and associative;
- is continuous;
- a * 1 = a for all a ∈ [0,1];
- a * b ≤ c * d whenever a ≤ c and b ≤ d for all a, b, c, d ∈ [0, 1].

Examples of continuous t-norms are Lukasiewicz t-norm, that is, a * b = max{a + b − 1, 0}, product t-norm, that is, a * b = ab, and minimum t-norm, that is, a * b = min{a, b}.

B. Definition 2 [31]
A fuzzy metric space is a triple (X; M; *), where X is a non-empty set, * is a continuous t-norm and M is a fuzzy set on X × X × [0, +∞), satisfying the following properties:
(KM-1) M(x; y; 0) = 0 for all x; y ∈ X;
(KM-2) M(x; y; t) = 1 for all t > 0 if x = y;
(KM-3) M(x; y; t) = M(y; x; t) for all x; y ∈ X and for all t > 0;
(KM-4) M(x; y; t) : [0; +∞) → [0; 1] is left continuous for all x,y ∈ X;
(KM-5) M(x; z; t+s) ≥ M(x; y; t) * M(y; z; s) for all x, y, z ∈ X and for all t, s > 0.
We denote such space as KM-fuzzy metric space.

C. Definition 3[32]
Let (X, M, *) be fuzzy metric space then
1) A sequence {x_n} in X is said to be Cauchy sequence if for all t > 0 and p>0.
\lim_{n→∞} M(x_{n+p}, x_n, t) = 1 And
2) A sequence {x_n} in X is said to be convergent to a point x ∈ X if for all t > 0, \lim_{n→∞} M(x_{n}, x, t) = 1.

D. Definition 4 [32]
A fuzzy metric space (X, M, *) is said to be complete if and only if every Cauchy sequence in X is convergent.
E. Definition 5
A pair of self-mapping (A, S) of a fuzzy metric space (X, M, *) is said to be commuting if M(ASx, SAx, t) = 1 for all x in X.

F. Definition 6
A pair of self-mapping (A, S) of a fuzzy metric space (X, M, *) is said to be weakly commuting if M(ASx, SAx, t) ≥ M(Ax, Sx, t) for all x in X and t > 0.

In 1994, Mishra et al. [33] introduced the concept of compatible mapping in fuzzy metric space to concept of compatible mapping in metric space as follows:

G. Definition 7 [33]
A pair of self-mapping (A, S) of a fuzzy metric space (X, M, *) is said to be compatible if lim M(ASx, SAx, t) = 1 for all t > 0, whenever {x_n} is a sequence in X such that lim A x_n = lim S x_n = z for some u in X.

H. Definition 8
Let (X, M, *) be a fuzzy metric space A and S be self maps on X. A point x in X is called a coincidence point of A and S iff Ax = Sx. In this case, w = Ax = Sx is called a point of coincidence of A and S.

I. Definition 9 [34]
A pair of self-mapping (A, S) of a fuzzy metric space (X, M, *) is said to be weakly compatible if they commute at the coincidence points. i.e., if Az = Sz for some z ∈ X then ASz = SAz.

It is easy to see that two compatible maps are weakly compatible but converse is not true.

J. Definition 10 [35]
A pairs (A, S) of self mapping of a fuzzy metric space (X, M, *) is said to satisfying the property (E.A) if there exist a sequence {x_n} in X such that lim A x_n = lim S x_n = z for some z ∈ X.

K. Definition 11 [35]
Two pairs (A, S) and (B, T) of self mappings of a KM- (or GV-) fuzzy metric space (X, (\cdot)) are said to satisfy the common property (E.A) if there exist two sequences {x_n}, {y_n} in X such that for all t > 0, lim A x_n = lim S x_n = lim B y_n = lim T y_n = z, for some z ∈ X.

L. Definition 12 [36]
A pair of self-mappings (A, S) of a fuzzy metric space (X; M; (\cdot)) is said to satisfy the common limit in the range of g property (CLRRS) if there exists a sequence {x_n} in X such that lim A x_n = lim S x_n = z for some z ∈ X.

With a view to extend the (CLRg) property to two pair of self mappings, very recently Imdad et. al. [37] defines the (CLRST) property (with respect to mappings S and T) as follows:

M. Definition 13 [37]
Two pairs (A; S) and (B; T) of self mappings of a fuzzy metric space (X; M; (\cdot)) are said to satisfy the (CLRST) property (with respect to mappings S and T) if there exist two sequences {x_n}, {y_n} in X such that, lim A x_n = lim S x_n = lim B y_n = lim T y_n = z, for some z ∈ S(X) ∩ T(X).

N. Lemma 2.1 [32]
Let (X, M, *) be a fuzzy metric space Then M(x, y, t) is non-decreasing for all x, y ∈ X.

O. Lemma 2.2 [16]
Let (X, M, *) be a fuzzy metric space if there exist k ∈ (0, 1) such that M(x, y, kt) ≥ M(x, y, t) for all x, y ∈ X and t > 0, then x = y.

III. MAIN RESULTS
In this section, we first introduce the notion of “the joint common limit in the range property” of two pairs of self mappings.

A. Definition 3.1
Let (X, M, *) be a fuzzy metric space and A, B, S, T: X → X. The pair (A, S) and (B, T) are Said to satisfy the “joint common limit in the range of S and T” property (shortly, (JCLRST) property) if there exists a sequence {x_n} and {y_n} in X such that

\lim_{n \to \infty} A x_n = \lim_{n \to \infty} S x_n = \lim_{n \to \infty} B y_n = \lim_{n \to \infty} T y_n = z, for some u ∈ X.

Our results involve the class of all functions \(\phi: [0, 1] \to [0, 1]\) satisfying the following properties:

- \(\phi\) is continuous and non-decreasing on \([0, 1]\).
- \(\phi(s) > s\) for all \(s \in (0, 1]\).
- \(\phi(1) = 1\).
- \(\phi(s) \geq s\) for all \(s \in (0, 1]\).

Now we start with the following theorem:

B. Theorem 3.1
Let (X, M, *) be a fuzzy metric space where * is a continuous t-norm and A, B, S, T be mapping from X into itself further let the pair (A, S) and (B, T) are weakly compatible and there exists a constant \(k \in (0, 1]\) such that

\[ M(Ax, By, kt^2) \geq \min \left\{ M(St, Ty, t^2), M(Sx, Ax, t), M(Ty, By, t^2), M(Ty, Ax, t) \right\} \]

(1)

Holds for all x, y ∈ X, & t > 0 and \(\phi \in \Phi\). if \((A, S)\) and \((B, T)\) satisfy the (JCLRST) property then A, B, S, T have a unique common fixed point in X.

1) Proof
Since the pairs (A, S) and (B, T) satisfy the (JCLRST) property there exists a sequence \(\{x_n\}\) and \(\{y_n\}\) in X such that

\[ \lim_{n \to \infty} A x_n = \lim_{n \to \infty} S x_n = \lim_{n \to \infty} B y_n = \lim_{n \to \infty} T y_n = z, \text{ for some } u \in X. \]

Now we assert that Bu = Tu. using (1), with x = x_n and y = u, we get

\[ M(Ax_n, Bu, kt^2) \geq \min \left\{ M(Sx_n, Tu, t^2), M(Sx_n, Ax_n, t), M(Tu, Bu, t^2), M(Tu, Ax_n, t) \right\}. \]

Taking the limit as \(n \to \infty\), by using JCLR property we have
Let $(X, M, *)$ be a fuzzy metric space if there exists a fuzzy metric $d$ such that $d(x,y) = M(x,y,t)$ for all $x,y \in X$ and $t \in [0,1]$. Then $(X, d)$ is a complete metric space.

**Corollary 3.1**

Let $(X, M, *)$ be a fuzzy metric space where $*$ is a continuous t-norm and $A, S, T$ be mappings from $X$ into itself. Then for $x, y \in X$, we have

$$d(Ax, Ay) = M(x, y, t) \leq M(\min\{M(x, z, t), M(y, z, t)\}, M(z, z, t))$$

for all $x, y, z \in X$ and $t \in [0,1]$. This implies that $A, S, T$ have a unique common fixed point in $X$.

**Corollary 3.2**

Let $(X, M, *)$ be a fuzzy metric space where $*$ is a continuous t-norm and $A, S, T$ be mappings from $X$ into itself. Further, let the pair $(A, S)$ is weakly compatible and there exist a constant $k \in (0, \frac{1}{2})$ such that

$$M(\min\{M(Tx, Ay, t), M(Ty, Ax, t)\}, M(Tx, Ty, t)) \leq k M(Tx, Ty, t)$$

holds for all $x, y \in X$, $t > 0$ and $k \in (0, \frac{1}{2})$. Then $A, S, T$ have a unique common fixed point in $X$.

**Theorem 3.2**

Let $(X, M, *)$ be a fuzzy metric space where $*$ is a continuous t-norm. Further, let the pair $(A, S)$ be self-mapping and there exist a constant $k \in (0, \frac{1}{2})$ such that

$$M(\min\{M(Tx, Ay, t), M(Ty, Ax, t)\}, M(Tx, Ty, t)) \leq k M(Tx, Ty, t)$$

holds for all $x, y \in X$, $t > 0$ and $k \in (0, \frac{1}{2})$. Then $A, S, T$ have a unique common fixed point in $X$.

**IV. INTEGRAL ANALOGUE OF RELATED FIXED POINT THEOREMS**

Branciari[38] firstly states and prove an integral -type fixed point theorem which generalized the well-known Banach Contraction Principle since then, many researchers have extensively proved several common fixed point theorems satisfying integral type contractive condition [39,40,44]; in this section, we state and prove an integral analogue of theorem 3.1.

**A. Lemma**

Let $(X, M, *)$ be a KM-fuzzy metric space if there exists a constant $k \in (0, \frac{1}{2})$ such that for all $x, y \in X$ and all $t > 0$, we have

$$M(\max\{M(x, y, t), M(y, x, t)\}, M(x, x, t), M(y, y, t)) \leq k M(x, y, t)$$

holds for all $x, y \in X$, $t > 0$, and $k \in (0, \frac{1}{2})$. Then $A, S, T$ have a unique common fixed point in $X$.
Where,
\[
m(u,y_n) = \min \left( M(Su,Ty_n,t)^2, M(Su,Au,t), M(Ty_n,By_n,t), M(Su,By_n,2t), M(Ty_n,Au,t) \right)
\]

Taking the limit as \( n \to \infty \), and using JCLR property we have
\[
\lim_{n \to \infty} M(Au,Bu,y_n,kt)^2 \geq \int_0^\infty \varphi(s) ds \geq \phi \left( \int_0^\infty \varphi(s) ds \right)
\]

by lemma 2.2 we have \( Au=Tu \).

Now we assume that \( z=Au=Bu=Su=Tu \) since the pairs \((A,S)\) is a weakly compatible \( AS=SA \) and then \( Az=AS=AU=SZ \), it follows from \((B,T)\) is a weakly compatible \( BT=BU \) and hence \( Bz=BT=BU=TZ \).

we show that \( z=Bz \). To prove this using (4) with \( x=z, y=u \), we get
\[
\lim_{n \to \infty} M(z,u)^2 \geq \int_0^\infty \varphi(s) ds \geq \phi \left( \int_0^\infty \varphi(s) ds \right)
\]

which is implies \( Az=z \) hence \( z=AZ=SZ \).

Next we show that \( Au=Tu \) using (4) with \( x=u \), \( y=z \) we get
\[
\lim_{n \to \infty} M(u,z)^2 \geq \int_0^\infty \varphi(s) ds \geq \phi \left( \int_0^\infty \varphi(s) ds \right)
\]

where,
\[
m(u,z) = \min \left( M(Su,Tz,t)^2, M(Su,Au,t), M(Tz,Bz,2t), M(Su,Bz,2t), M(Tu,Au,t) \right)
\]
Where
\[ m(u, z) = \min\left( \frac{M(z, Bz, t)^2}{M(Bz, Bz, t), M(z, Bz, 2t), M(Bz, z, t)} \right) \]

We get
\[ \int_0^t \varphi(s) ds \geq \left( \int_0^t \varphi(s) ds \right) \]

Which implies \( Bz = z \), hence \( z = Bz = TZ \). Therefore we conclude that \( z = A_z = Bz = S_z = Tz \) if this implies \( A, B, S, T \) have common fixed point that is a point \( z \). uniqueness of the common fixed point is an consequence of condition (4) in respect of condition \((\varphi)\). This concludes the proof.

REFERENCES