Comparison of Direct and Iterative Methods of Solving System of Linear Equations

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Abstract— The paper presents a Survey of a direct method and two Iterative methods used to solve system of linear equations. An example of system of linear equations is taken and solved manually. The result shows that which method is more efficient out of the three, considering their performance, using parameters like time to converge, number of iterations required to converge, storage and level of accuracy. This research will enable the persons to appreciate the use of most easy method to solve the system of linear equations which takes less time to solve the system of linear equations.

Key words: Iterative Methods, LU Method, The Gauss Seidel Method

I. INTRODUCTION

In mathematics the theory of system of linear equations is the basis and fundamental part of the Linear Algebra, a subject used in most part of modern mathematics. Computational algorithms to find solutions are an important part of numerical linear algebra, and plays important role in engineering, physics, chemistry, computer science and economics. Even a system of non-linear equations could be approximated by a linear system.

A linear equation is an algebraic equation in which each term is either a constant or the product of a constant and (the first power of) a single variable.

And system of such equations is known as system of linear equations. The most common type of problem is to solve a square linear system of type

\[ Ax = b \]

Of moderate order with coefficient that are mostly non-zero, such linear system of any order are dense since the coefficient matrix A is generally stored in the main memory of the computer in order to efficiently solve the linear system, and storage limitations in most of the computers limit the system to have iterations less than 100 to 200 depending on the computer

To solve such system of equations we have many methods which are basically divided into subcategories of direct and indirect methods. The efficiency of any method is judged by two criteria:

1) How fast the method is in solving the system of equations. That is number of iterations involved in the method.

2) Accuracy of the same method by using computers. The need of first point is to minimize the time of calculation in solving the system of equations. And the need of second point arise because small round off errors may occur during calculation part which may cause errors in the computer solution out of all proportion to their size, or in large calculations during calculative part potential round off errors may cause substantial loss of accuracy.

Generally the coefficient matrices occurring in practice fall in either of the two following categories:

1) Filled but not large: This means that the matrix is of order less than 100. Such matrices occur in problems of engineering, statistics, etc.

2) Sparse but very large: Matrix of very large order and having very few non-zero elements like matrix of order one thousand, such matrices are called sparse matrices. And such matrices occur commonly in numerical solution of partial differential equations.

Direct methods are generally used to solve problems falling in first category. In direct method values of \( x_i \)'s are found by directly solving the system of equations. For the second category iterative methods are generally used since the direct methods are known to have their difficulties. For example in Gauss elimination method problem is to control the accumulation of rounding errors

A. Direct Methods:

These methods theoretically gives an exact solution of the system of equations in a finite number of steps. But in practices this may not hold true.

B. Iterative Methods:

Using approximate methods for solving system of linear equations we can obtain the values of the roots of the system or solution of the system with the specified accuracy as the limit of the sequence of some vectors. Process of constructing such a sequence is called an iteration.

In this work one direct method and two iterative methods are studied. In direct method an exact solution can be obtained in a finite number of operations. While in iterative methods we start with some initial approximation and generate successively improved approximations in an infinite sequence whose limit is the exact solution.

C. Gauss Elimination Method (Direct Method):

Another direct method of solving the system of linear equations. Also known as row reduction method.

D. Procedure:

The system of equations is first converted to matrix form of type \( AX = b \), then taking augmented matrix \([A|b]\), row elementary operations are performed on both \( A \) and \( b \) so that the coefficient matrix converts to upper triangular matrix. After that by applying back substitution one can get the solutions of the system of linear equations.

For example consider a system of linear equations of three variables say:

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
    a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{align*}
\]

The augmented matrix of the given system of equation is
\[ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \]

Where 
\[ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \]
is the coefficient matrix.

By applying elementary row operations and converting the coefficient matrix to upper triangular matrix, one will get the following form:
\[ \begin{bmatrix} a'_{11} & a'_{12} & a'_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a'_{33} \end{bmatrix} \]

Then by applying back substitution in above matrix starting from the third row then second then first, one can calculate the values of \( x_1, x_2 \) and \( x_3 \).

E. LU Method (Direct method):

Introduced by Mathematician Alan Tango this LU method is used to solve the system of linear equations. The first step is to convert the given system to matrix form. And then after using this method the matrix is decomposed into the product of lower and upper triangular matrix. It can be viewed as a matrix form of gauss elimination.

F. Procedure:

Let \( A \) be a square matrix. An LU factorization refers to the factorization of \( A \), with proper row and/or column orderings or permutations, into two factors, a lower triangular matrix \( L \) and an upper triangular matrix \( U \).

\[ A = LU \]

In the lower triangular matrix all elements above the diagonal are zero, in the upper triangular matrix, all the elements below the diagonal are zero. For example, for a 3-by-3 matrix \( A \), its LU decomposition looks like this:
\[ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \]

Without a proper ordering or permutations in the matrix, the factorization may fail to materialize. For example, it is easy to verify (by expanding the matrix multiplication) that \( a_{11} = l_{11}u_{11} \). If \( a_{11} = 0 \) then at least one of \( l_{11} \) and \( u_{11} \) has to be zero, which implies either \( L \) or \( U \) is singular. This is impossible if \( A \) is nonsingular. This is a procedural problem. It can be removed by simply reordering the rows of \( A \) so that the first element of the permuted matrix is nonzero. The same problem in subsequent factorization steps can be removed the same way.

It turns out that a proper permutation in rows (or columns) is sufficient for the LU factorization.

G. The Gauss Seidel Method (Iterative method):

Consider again the linear equation in (1)
\[ \sum_{j=1}^{n} a_{ij}x_j = b_j \]

To find the value of \( x_j \) by assuming that the other entries of \( x \) remain fixed, we have
\[ x^{(k)}_j = \left[ b_j - \sum_{i \neq j} a_{ij}x^{(k)}_i - \sum_{i \neq j} a_{ij}x^{(k-1)}_i \right] / a_{jj} \]

The calculation in each iteration should be done properly because the value in each iteration depends upon the value of previous iteration and the solution of the equation depends on the values in iteration. It is often called method of successive displacements as it indicates the dependency of the iterates on the order. Hence if there is any change in the order the components of the new iterate will also change.

H. Successive Relaxation method (Iterative method):

This method is obtained by applying extrapolation to the Gauss-Seidel method. For each component this extrapolation takes the form of a weighted average between the previous iterate and the computed Gauss-Seidel iterate successively.
\[ x_i^{(k)} = \left[ \omega x_i^{(k)} + (1 - \omega) x_i^{(k-1)} \right] \]

Where \( x_i \) denotes the \( i^{th} \) iterate
And \( \omega \) represents the extrapolation factor.

The idea is to choose a value of \( \omega \) which will accelerate the rate of convergence of iterates to the solution.

Illustration and computation: To compare the direct and iterative methods to solve a system of linear equations consider a system of linear equation:
\[ \begin{align*}
20 - 5l_1 - 3(l_2 - l_3) &= 0 \\
-3(l_1 - l_2) - (l_2 - l_3) &= 0 \\
-(l_1 - l_2) - 2l_2 &= 0
\end{align*} \]

Results produced by the equation using the direct and iterative methods are given in the table below:

<table>
<thead>
<tr>
<th>Methods</th>
<th>Value of ( l_1 )</th>
<th>Value of ( l_2 )</th>
<th>Value of ( l_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauss Elimination</td>
<td>3.114791</td>
<td>1.639442</td>
<td>-2.786885</td>
</tr>
<tr>
<td>L U Method</td>
<td>3.114754</td>
<td>1.639344</td>
<td>-2.786885</td>
</tr>
<tr>
<td>Gauss Seidel</td>
<td>3.114973</td>
<td>1.63958</td>
<td>-2.78681</td>
</tr>
<tr>
<td>Successive Relaxation</td>
<td>3.058</td>
<td>1.483</td>
<td>-2.8388</td>
</tr>
</tbody>
</table>

Table 1:

II. CONCLUSION

A. Comparison between Direct Methods:

While comparing between the direct methods it is observed from the taken example that from the two direct methods Gauss elimination and LU factorization, the later one takes less time to solve the given system of linear equation.

B. Comparison between Iterative Methods:

Comparing the two iterative methods Gauss Seidel and relaxation methods it is observed that time taken by the later one is less than the previous one.

C. Comparison between Direct and Iterative Methods:

Direct method gives direct values of the variables which cannot be changed, and is easy to calculate the solution within short time period. While in case of iterative methods time taken to solve the system of equations is comparatively more than time taken in direct methods. Final values of the solution and number of iteration depends on the Initial value taken to start the iteration. In these methods we approach
closer to the exact solution as much as iterations are done that is the more the number of iterations the more accurate will be the solution.

REFERENCES


