Relative Stability Analysis of Linear Systems based on Damping Factor

Sreekala K[1] S. N. Sivanandam[2]
[1,2]Professor
[1]EEE, SNGCE, Kadayiruppu [2]Emeritus, KIT, Coimbatore

Abstract— Information about the relative stability of a control system is of paramount importance for any design problem. In this paper two algebraic criteria for the relative stability analysis of linear time-invariant systems are formulated. When the relative stability analysis is done based on the damping ratio, characteristic equations with complex coefficients arise. These complex coefficients are used in two different ways to form the Modified Routh's tables for the two schemes named as Sign Pair Criterion I and Sign Pair Criterion II It is found that the proposed algorithms offer computational simplicity compared to other algebraic methods and is illustrated with suitable examples.

Keywords: damping ratio, characteristic equations, algebraic methods

I. INTRODUCTION

Once the system is found to be stable, it is important to determine how stable it is and this degree of stability is a measure of relative stability. Relative stability analysis can be done based on damping margin and damped frequency of oscillation in time domain analysis as presented by Hwang &Tripathi [1]. In frequency domain, the analysis is done based on gain margin and phase margin as discussed in Nagrath & Gopal [2].When the relative stability analysis is done in time domain, characteristic equations with complex coefficients arise. Then the normal Routh's algorithm cannot be applied.

To analyse the stability of complex polynomials the generalized Routh-Hurwitz method was investigated in [3] - [7]. Frank [3] and Agashe [4] developed a new Routh like algorithm to determine the number of RHP roots in the complex case. Benidir and Picinbono [5] proposed an extended Routh table which considers singular cases of vanishing leading array element. By adding intermediate rows in the Routh array, Shyan and Jason [6] developed a tabular column ,which is also a complicated one. Adel [7] has done the stability analysis of complex polynomials using the J-fraction expansion , Hurwitz Matrix determinant and also generalized Routh's Array.

Hwang & Tripathi [1] suggested an algebraic method using complex conjugates to convert complex coefficient equations to real coefficient equations for relative stability analysis The same method was proposed for the analysis of relative damping margin in the literature [8] which is a complicated one for computation.

Here the complex coefficients are used in two different ways to form the Modified Routh's tables for the two schemes named as Sign Pair Criterion I (SPC I) and Sign Pair Criterion II (SPC II). The beauty of the Routh's algorithm lies in finding the relative stability of the system without determining the roots of the system.

The first approach, formation of Routh's Table is done by retaining the 'j' terms of the complex coefficients and the stability analysis is done using Sign Pair Criterion I (SPC I) . The proof is given in [9].

The second scheme, a geometrical procedure is presented which is named as Sign Pair Criterion II (SPC II) and is formulated with the help of 'Modified Routh's table' after separating the real and imaginary parts of the characteristic equation by substituting s='jω'. Applying Routh–Hurwitz criterion, the number of the roots of F(s)=0 having positive real part can be revealed. The proof for SPC II is given in [10]. Then by the use of the proposed schemes relative stability is analyzed in a most simple way regardless of the order of the system .The computational simplicity is illustrated with examples.

II. PROPOSED SCHEMES

A. *Sign Pair Criterion I (SPC I):*

With all the coefficients positive, the characteristic equation C(s) can be written as,

$$C(s) = s^n + (a_1 + j\,b_1)s^{n-1} + (a_2 + j\,b_2)s^{n-2} + \cdots + (a_k + j\,b_k) = 0$$

The first two rows of Routh-like table are written a shown below :)

1	jb_1	a_2	jb_3	a_4	. . .
a_1	jb_2	a_3	jb_4	a_5	. . .

Applying the standard Routh multiplication rule [2], the subsequent elements of Routh-like table are computed and the table is computed as given below:

1	jb_1	a_2	jb_3	a_4	. . .
a_1	jb_2	a_3	jb_4	a_5	. . .
r_{31}	r_{32}	r_{34}	r_{35}	...	
r_{41}	r_{42}	r_{43}	r_{44}	...	
r_{51}	r_{52}	r_{53}	...		
r_{61}	r_{62}	r_{63}	...		
r_{71}	r_{72}	...			
r_{81}	...				
.	.	.	.		
.	.	.	.		

Using the first column elements, sign pairs are formed as

$$P_1 = (1, a_1), \quad P_2 = (r_{31}, r_{41}) \quad P_3 = (r_{51}, r_6$$
$$P_4 = (r_{71}, r_{81}) \; . \; . \; .$$

According to the first scheme SPC I, it is ascertained that each element of all the pairs has to maintain the same sign for the roots of characteristic equation to lie on the left hand side of s-plane for stability. The proof of the criterion is given in [9].

B. *Sign Pair Criterion II (SPC II):*

In this paper, another scheme is proposed for the analysis of stability of a given linear time - invariant system. With the substitution of s = jω $s =$., the real and imaginary parts of the characteristic equations $C(j\omega) =$, are extracted separately and their coefficients are entered suitably in the first-two rows of Routh-like table to observe the system stability. The formulated stability criterion is termed as 'Sign Pair Criterion-II' (SPC-II). In this procedure, it can be noted that the Routh-like table

Contains only real elements. Let

$$C(s) = s^n + (a_1 + j\,b_1)s^{n-1} + (a_2 + j\,b_2)s^{n-2} + \cdots + (a_k + j\,b_k) = 0$$

Substituting of s = jω ,

$$C(j\omega) = (j\omega)^n + (a_1 + j\,b_1)(j\omega)^{n-1} + (a_2 + j\,b_2)(j\omega)^{n-2} +$$
$$\ldots + (a_k + jb_k) = 0 \ldots + (a_k + jb_k) = 0$$
$$= R(\omega) + jI(\omega) = 0$$
$$R(\omega) = (A_0\omega^n + A_1\,\omega^{n-1} + A_2\,\omega^{n-2} + \cdots + A_n)$$
$$I(\omega) = (B_0\omega^n + B_1\,\omega^{n-1} + B_2\,\omega^{n-2} + \cdots + B_n)$$

Using the coefficients of above polynomials, the second form of Routh-like table can be formulated as

A_0	A_1	A_2	…	A_n
B_0	B_1	B_2	…	B_n
c_0	c_1	c_2	…	
d_0	d_1	d_2	…	
e_0	e_1	e_2	…	
f_0	f_1	f_2	…	
g_0	g_1	…	…	
.	.	.		
.	.	.		
.	.	.		

Algorithm for the proposed approach

1) If the first element in the first row is negative, multiply the full row elements by -1.
2) If the first element in the second row is zero, interchange first and second rows and multiply all elements in the second row by − −1.
3) Follow the Common Routh's multiplication rule to get the complete table with '2n+1' rows.
4) If any element of the first column starting from third, comes zero, it is replaced by a small value +0.01.
5) If all the elements in a row become zero, then the auxiliary polynomial is formed using the previous row elements and differentiated once; the coefficients of this modified polynomial are entered instead of zeros and the table is completed by applying the Routh multiplication rule.
6) Get 'n' sign pairs using the first column elements starting from second row.

The sign pairs are developed as

$$P_1 = (B_0, c_0)$$
$$P_2 = (d_0, e_0),\quad P_3 = (f_0, g_0)\ . \quad ..$$

According to the second scheme SPC II, it is ascertained that each element of all the pairs has to maintain the same sign for the roots of characteristic equation to lie on the left hand side of s-plane for stability. The proof of the criterion is given in [10].

III. RELATIVE STABILITY ANALYSIS

A. Analysis Based On Damping Ratio:

The transient response of a system having a good margin of absolute stability may possess unsatisfactory oscillations with poor damping ratio. These oscillations may be prevented by specifying the roots of the characteristic equation to lie in a sector region in the left-half of s -plane like that shown in Figure 1 [11].

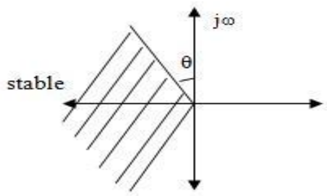

Fig. 1: Region of Relative Stability in the S-Plane

If all the roots of the characteristic equation are expected to lie on the marked region of Figure 1, then the transient response will have damping margin, greater than or equal to $\sin\theta$ or $\cos(90 -$ sin θ or cos (90-θ).This indicates that, greater the relative stability margin, the fewer will be the number of oscillations and they decay to a specified level from an initial position.

To investigate the above situation, the characteristic equation with positive real coefficients is transformed into another characteristic equation possessing complex coefficients. This is obtained by substituting $s = S\angle\theta\; s = S\angle\theta$, $s = S\angle$ where S is a new variable. Thus, the resulting equation,

$$C(S\angle\theta) = a_0(\cos n\theta + j\sin n\theta)S^n + a_1S^{n-1} +$$
$$\ldots + a_{n-1}(\cos\theta + j\sin\theta)S + a_n = 0$$

Where $\angle n\theta = e^{jn\theta} = \cos n\theta + j\sin n\theta$

It can be observed that the equation has complex coefficients and the proposed algebraic stability criteria SPC-I and SPC-II can easily be applied for investigation of stability.

B. Illustrations:

Example 1

Check whether the given characteristics equation has a damping ratio δ=0.5.

$$C(s) = s^3 + 4s^2 + 4s + 3 = 0$$

Since, δ= sin θ, θ=π/6

$$s = S\angle\frac{\pi}{6} = S(\cos 30^\circ + j\sin 30^\circ)$$
$$C\left(S\angle\frac{\pi}{6}\right) = (-0.5 + j0.867)S^3 + \left(2 + j\,2\sqrt{3}\right)S^2 +$$
$$\left(2\sqrt{3} + j2\right)S + 3 = 0$$
$$C(S) = S^3 + (2 - j3.464)S^2 + (0 - j4)S + (-1.5 - j2.6) = 0$$

Routh-like table as per SPC-I

$+1$	$-j3.464$	0	$-j2.6$
$+2$	$-j4$	-1.5	0
$-j1.464$	0.75	$-j2.6$	
$-j5$	-5		
$+2.22$	$-j2.6$		
$+0.83$			

From the first column, the three pairs are formed as

$P_1 = (+1, +2)$, $P_2 = (-j1.464, -j5)$ and $P_3 = (+2.22, +0.83)$

Since all the elements in each pair have the same sign, the above pairs satisfy SPC-I.

Application of SPC-II

By substituting s='jω' in the equation

$$C(j\omega) = R(\omega) + jI(\omega) = 0$$
$$= (-2 + 4\,\omega - 1.5) + j\left((-\omega^3 + 3.464\,\omega^2 - 2.6\right) = 0$$

Routh-like table as per SPC-II

0	−2	4	−1.5
−1	3.464	0	−2.6
−2	4	−1.5	
+1.464	0.75		
+5	−5		
+2.2	−2.6		
+0.83			

The three pairs formed from the above first column are

$P_1 = (-1, -2)$, $P_2 = (+1.464, +5)$ and $P_3 = (+2.2, +0.83)$.

Since all the elements in each pair have the same sign, the above pairs satisfy SPC-II.

Verification: The location of roots are as follows.

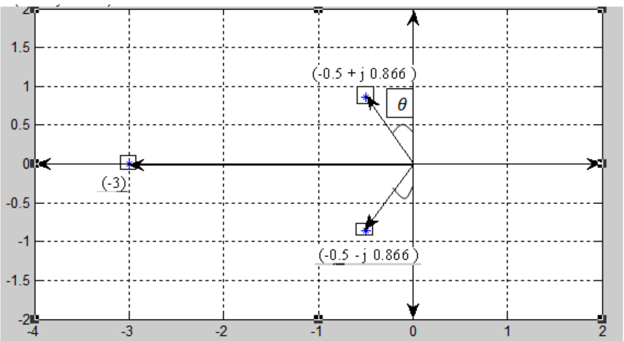

Fig. 2: Location of Roots

For the first root (-3),

$$\theta = 90, \quad \zeta = \cos(90 - \theta) = \cos 0 = 1$$

For the second root (-0.5 + j 0.866),

$$\theta = \tan^{-1}\left(\frac{0.5}{0.866}\right) = 30^0, \quad \zeta = \cos(90 - \theta)$$
$$= \cos 60 = 0.5$$

For the third root (-0.5 - j 0.866),

$$\theta = 180 - \tan^{-1}\left(\frac{0.5}{0.866}\right) = 150'$$
$$\theta = 180 - \tan^{-1}\left(\frac{0.5}{0.866}\right) = 150'$$
$$\zeta = \cos(90 - \theta) = \cos(-60) = 0.5$$

It is verified that all the roots have damping ratio greater than or equal to 0.5 and the application of SPC–I and SPC–II are found to be simpler than that given in [11].

IV. Conclusion

In this paper, the relative stability analysis of a time invariant continuous system has been performed with the help of the proposed SPC-I and SPC-II. The transient response behavior of a linear time-invariant continuous system having absolute stability has been carried-out based on damping ratio.The proposed algebraic criteria are simple and direct in application compared to other schemes .

REFERENCES

[1] Hwang, HH and Tripathi, PC, “Generalization of the Routh-Hurwitz criterion and its applications,” Electronics Letters, vol.6, no.13, pp.410–411,1970
[2] Nagrath, IJ and Gopal, M, Control Systems Engineering, New Age International (P) Ltd, New Delhi, India,2007
[3] Frank, E, “On the zeros of polynomials with complex coefficients”, Bulletin of American Mathematical society, vol.52, no.10, pp.144-158, 1946.
[4] Agashe, SD, “A new general Routh – like algorithm to determine the number of RHP Roots of a real or complex polynomial”, IEEE Transactions on Automatic Control vol.30, no.4, pp.406-409,1985.
[5] Bendir, M & Picinbono, B, “The extended Routh’s table with complex case”, IEEE Transactions on Automatic Control, vol.36, no.2, pp.253-256,1991.
[6] Shyan S. Chen and Jason S.H. Tsai, “A new Tabular form for determining root distribution of a complex polynomial with respect to the imaginary axis”, IEEE Transactions on Automatic Control, Vol.38, no.10, pp.1536-1541, 1993.
[7] Hashem, AMK, Network synthesis of complex Impedance and complex reactance functions, Ph.D. Thesis, The Department of Electrical and Computer Engineering, Concordia University, Montred , Canada, 1993.
[8] Usher, T, ‘A new application of the Hurwitz–Routh stability criteria’, Transactions of the AIEE, Part-I Communication and Electronics, vol.76, no.5, pp. 530-533, 1957.
[9] Sreekala K and S.N.Sivanandam, “An algebraic Approach for Stability Analysis of Linear Systems with Complex Coefficients”, International Journal of Computer Applications, Vol.44, No.3, pp.13-15, 2012.
[10] Modified Routh’s Table for the Stability Analysis of Linear Systems Having Complex Coefficient Polynomials”, International Review of Mechanical Engineering, Vol.6,No.6,pp.1213-1216,2012
[11] Porter, B, Stability Criteria for Linear Dynamical Systems, Oliver & Boyd. Edinburgh U.K, 1967.
[12] Bakshi,UA & Bakshi ,VU, Principles of Control Systems, Technical Publications Pune,2009