On a Result of Fixed Point Theorems for Commuting and Weakly Compatible Maps

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Abstract— Retaining the concept of complex valued metric spaces introduced by Azam et al.[1],considerable fixed point theorems have been proved using some mappings satisfying certain point-dependent contractive conditions. Rahul Tiwari et. al[2] proved common fixed point theorem with six maps in complex valued metric spaces And Dr. Yogita sharma [3] obtained common fixed point theorem for six maps which generalized and extended the results of Rahul Tiwari. In this paper we establish some common fixed point theorems for nine maps in complex valued metric spaces having commuting and weakly compatible and satisfying different types of inequality.

Key words: Commuting Mapping, Weakly Compatible Maps, and Common Fixed Points, Complex Valued Metric Spaces

I. INTRODUCTION

Fixed point theorems are substantial and admirable tool for confirming the Existence and uniqueness of the solutions to various mathematical models like differential, integral and partial differential equations and vibrational inequalities etc. The study of metric space expressed the most common important role to many fields both in pure and applied science [4]. Abounding researchers extended the notion of a metric space such as vector valued metric space of Perov [8], a cone metric spaces of Huang and Zhang [6], a modular metric spaces of Chistyakov [7], etc

Many authors generalized and extended Preliminaries

Let $\mathbb{C}$ be the set of all complex numbers, for $z_1, z_2 \in \mathbb{C}$, define a partial order $\precsim$ on $\mathbb{C}$ as follows;

$$z_1 \precsim z_2 \text{ iff } Re(z_1) \leq Re(z_2),\ Im(z_1) \leq Im(z_2)$$

It follows that

$$z_1 \precsim z_2$$

If one of the following conditions is satisfied:

$$Re(z_1)=Re(z_2),\ Im(z_1)<Im(z_2)$$
$$Re(z_1)<Re(z_2),\ Im(z_1)=Im(z_2)$$
$$Re(z_1)<Re(z_2),\ Im(z_1)<Im(z_2)$$
$$Re(z_1)=Re(z_2),\ Im(z_1)=Im(z_2)$$

In particular, we will write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one the above conditions is not satisfied and we will write $z_1 \prec z_2$ if only iii is satisfied. Note that

$$0 \precsim z_1 \lnsim z_2 \Rightarrow |z_1|<|z_2|,$$
$$z_1 \precsim z_2\ ,\ z_1 \prec z_2 \Rightarrow z_1 \prec z_3$$

A. Definition 1.1[1]

Let X be a nonempty set. A mapping $d: X \times X \to \mathbb{C}$ is called a complex valued metric on X if the following conditions are satisfied:

- $0 \precsim d(x,y)$ for all $x,y \in X$ and $d(x,y)=0 \Leftrightarrow x=y$.
- $d(x,y)=d(y,x)$ for all $x,y \in X$
- $d(x,y) \precsim d(x,z)+d(z,y)$ for all $x,y,z \in X$.

In this case, we say that (X,d) is a complex valued metric space.

B. Definition 1.2

Let $\mathbb{C}$ be a complex valued metric space,

- We say that a sequence $\{x_n\}$ is said to be a Cauchy sequence be a sequence in $x \in X$ If for every $c \in \mathbb{C}$, with $0 \prec c$ there is $n_0 \in \mathbb{N}$such that for all $n>n_0$ such thatd$(x_n,x_m) \prec c$.
- We say that a sequence $\{x_n\}$ converges to an element $x \in X$. If for every $c \in \mathbb{C}$, with $0 \prec c$ ther exist an integer $n_0 \in \mathbb{N}$such that for all $n>n_0$ such that $d(x_n,x) \prec c$ and we write $x_n \xrightarrow[d]{} x$.
- We say that (x,d) is complete if every Cauchy sequence in X converges to a point in X.

C. Lemma 1.3

Any sequence $\{x_n\}$ in complex valued metric space (X,d), converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$

D. Lemma 1.4

Any sequence $\{x_n\}$ in complex valued metric space (X,d) is a Cauchy sequence if and only if $|d(x_n,x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$

E. Definition 1.5

Let S and T be self-maps of a non-empty set X. then

- Any point $x \in X$ is said to be fixed point of T if Tx=x.
- Any point $x \in X$ is said to be coincidence point of T and S if Sx=Tx and we shall called w=Sx=Tx that a point of coincidence of s and T.
- Any point $x \in X$ is said to be fixed point of T and S if Sx=Tx=x

F. Definition 1.6[10]

Let S and T be self-maps of metric space (x,d), then S,T are said to be weakly commuting if $d(STx,TSx) \leq d(Sx,Tx)$, for all $x \in X$

G. Definition 1.7

Let S, T be self-maps of metric space (x,d), then S,T are said to be compatible if $\lim_{n\to\infty} d(STx, TSx_n)=0$

Whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} S x_n = \lim_{n\to\infty} T x_n = z$, for some $z \in X$

H. Remark 1.8

In general, commuting maps are weakly commuting and weakly commuting maps are compatible, but converse are not necessarily true and some examples can be found in [8-10]

I. Definition 1.9

Two self-maps s,T of a non-empty set X are said to be weakly compatible is STx=TSx whenever sx=Tx.

J. Definition 1.10[5]

Two self-maps s,T of a non-empty set X are said to be commuting is TSx=STx for all x∈ X

K. Definition 1.11[9]

Let $T:X \to X$ be a map, then there exists a subset E of X such T€=T(X) and T: $E \to X$ is one to one.
Theorem: let (X,d) be a complex valued metric space and M,P,Q,R,S,T,U,V,W be self-maps of X satisfying the following condition

1) *TU⊆P(X) and RS⊆Q(X) and VW⊆M(X)*

d(RSx,TUy)+d(Tux,VWy)≤Ad(Mx,Py)+Bd(Px,Qy)+Cd(Qx,My)+D[d(Mx,RSx)+d(Py,TUy)]+E[d(Px,TUx)+d(Qy,VWy)]+F[d(Qx,RSx)+d(My,TUy)]+G[d(Qx,TUx)+d(My,VWy)]+H[d(Mx,TUy)+d(Py,RSx)]+I[d(Px,VWy)+d(Qy,TUx)]+J[d(Qx,TUy)+d(RSx,My)]

2) *+K[d(Qx,VWy)+d(TUx,My)]*

For all x,y,z∈ X where A,B,C,D,E,F,G,H,I,J,K ≥0 and A+B+C+D+E+F+G+H+I+J+K<1

Assume that the pairs (TU,Q) (RS,P) and (VW,M) are weakly compatible. Pairs (T,U) (T,Q) (Q,U) (R,S) (R,P) (P,S) (V,W (V,M)(M,V) are commuting pairs of maps. Then M,P,Q,R,S,T,U,V,W have a unique common fixed point in X.

Proof: Let x_0∈ X. by (1.1) we can define inductively a sequence $\{y_n\}$ and $\{z_n\}$ in X such that $y_{2n}=RSx_{2n}=TUx_{2n+1}=Px_{2n+1}=Px_{2n+1}==VWx_{2n+1}=Mx_{2n+1}$,

3) *For all n= 0,1,2,3…*

$d(RSx_{2n},TUx_{2n+1})+d(Tux_{2n},VWx_{2n+1})\le Ad(Mx_{2n},Px_{2n+1})$
$+Bd(Px_{2n},Qx_{2n+1})$
$+Cd(Qx_{2n},Mx_{2n+1})$
$+D[d(Mx_{2n},RSx_{2n})+d(Py,TUx_{2n+1})]$
$+E[d(Px_{2n},TUx_{2n})+d(Qx_{2n+1},VWx_{2n+1})]$
$+F[d(Qx_{2n},RSx_{2n})+d(Mx_{2n+1},TUx_{2n+1})]$
$+G[d(Qx_{2n},TUx_{2n})+d(Mx_{2n+1},VWx_{2n+1})]$
$+H[d(Mx_{2n},TUx_{2n+1})+d(Px_{2n+1},RSx_{2n})]$
$+I[d(Px_{2n},VWx_{2n+1})+d(Qx_{2n+1},TUx_{2n})]$
$+J[d(Qx_{2n},TUx_{2n+1})+d(RSx_{2n},Mx_{2n+1})]$
$+K[d(Qx_{2n},VWx_{2n+1})+d(TUx_{2n},Mx_{2n+1})]$
$d(y_{2n},y_{2n+1})+d(y_{2n},y_{2n+1}))\le Ad(y_{2n-1},y_{2n})$
$+Bd(y_{2n-1},y_{2n+1})$
$+Cd(y_{2n},y_{2n})+D[d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n})]$
$+E[d(y_{2n-1},y_{2n-1})+d(y_{2n},y_{2n})]$
$+F[d(y_{2n},y_{2n})+d(y_{2n},y_{2n})]$
$+G[d(y_{2n},y_{2n-1})+d(y_{2n},y_{2n})]$
$+H[d(y_{2n-1},y_{2n})+ d(y_{2n},y_{2n})]$
$+I[d(y_{2n-1},y_{2n+1})+d(y_{2n+1},y_{2n-1})]$
$+J[d(y_{2n},y_{2n})+d(y_{2n},y_{2n})]$
$+K[d(y_{2n},y_{2n+1})] + d(y_{2n-1},y_{2n})]$
$2d(y_{2n},y_{2n+1})\le Ad(y_{2n-1},y_{2n})+Bd(y_{2n-1},y_{2n+1})+D[d(y_{2n-1},y_{2n})]+G[d(y_{2n},y_{2n-1})] +H[d(y_{2n-1},y_{2n})]$
$+I[d(y_{2n-1},y_{2n+1})+d(y_{2n+1},y_{2n-1})]$
$+K[d(y_{2n-1},y_{2n+1})]2d(y_{2n},y_{2n+1})\le Ad(y_{2n-1},y_{2n})+Bd(y_{2n},y_{2n+1})-Bd(y_{2n-1},y_{2n})+D[d(y_{2n-1},y_{2n})]$
$+G[d(y_{2n},y_{2n-1})] +H[d(y_{2n-1},y_{2n})]$
$+K[d(y_{2n-1},y_{2n})]+ K[d(y_{2n},y_{2n+1})]$
$+I[2d(y_{2n-1},y_{2n})]+ 2I[d(y_{2n},y_{2n+1})]$
$2d(y_{2n},y_{2n+1}) - 2I[d(y_{2n},y_{2n+1})]-K[d(y_{2n},y_{2n+1})]\le Ad(y_{2n-1},y_{2n})+Bd(y_{2n-1},y_{2n+1})+D[d(y_{2n-1},y_{2n})]$
$+G[d(y_{2n},y_{2n-1})]+H[d(y_{2n-1},y_{2n})]$
$+I[2d(y_{2n-1},y_{2n})] + K[d(y_{2n-1},y_{2n})]$
$[d(y_{2n},y_{2n+1})(2-2I-K-B)\le$
$Ad(y_{2n-1},y_{2n})+Bd(y_{2n-1},y_{2n})+D[d(y_{2n-1},y_{2n})]$
$+G[d(y_{2n},y_{2n-1})]+H[d(y_{2n-1},y_{2n})]$
$+I[2d(y_{2n-1},y_{2n})] + K[d(y_{2n-1},y_{2n})]$
$d(y_{2n},y_{2n+1}) \le \frac{(A+B+D+G+H+2I+k)}{(2-2I-K-B)}d(y_{2n-1},y_{2n})$
$d(y_{2n},y_{2n+1})\le k\, d(y_{2n-1},y_{2n})$ where $k=\frac{(A+B+D+G+H+2I+k)}{(2-2I-K-B)}$
$d(y_{2n+1},y_{2n+2})\le k\, d(y_{2n},y_{2n+1}) \le k^2\, d(y_{2n-1},y_{2n})$
Therefore,
$d(y_n,y_m) \le d(y_n,y_{n+1})+ d(y_n,y_{n+1})+ d(y_{n+1},y_{n+2})+\dots d(y_{m-1},y_n)$
$\le (k^n+k^{n+1}+\dots+k^{m-1})d(y_1,y_0)$
$\le \frac{k^n}{1-k}\, d(y_1,y_0)$
$|d(y_n,y_m)| \le \frac{k^n}{1-k}|d(y_1,y_0)|$

Which implies that$|d(y_n,y_m)|\to 0$ as $n,m\to \infty$. hence $\{y_n\}$ is a Cauchy sequence.
Since X is complete, there exist a point z in X such that

4) $\lim_{n\to\infty} RSx_{2n} = \lim_{n\to\infty} Qx_{2n+1} = \lim_{n\to\infty} TUx_{2n+1} = \lim_{n\to\infty} Px_{2n+1} = \lim_{n\to\infty} VWx_{2n+1} = \lim_{n\to\infty} Mx_{2n+1} = z$

TU(X)⊆P(X) there exist
u∈ X such that z = Pu, then by equation (1.2)
$d(RSu,z)+d(TUu,z)=d(RSu,TUx_{2n+1})+d(TUx_{2n+1},z)+d(TUu,VWx_{2n+1})+d(VWx_{2n+1},z)$
$\le Ad(Mx_{2n},Px_{2n+1})+Bd(Pu,Qx_{2n+1})+Cd(Qx_{2n},Mx_{2n+1})$
$+D[d(Mx_{2n},RSu)+d(Px_{2n+1},TUx_{2n+1})]$
$+E[d(Pu,TUu)+d(Qx_{2n+1},VWx_{2n+1})]$
$+F[d(Qx_{2n},RSu)+d(Mx_{2n+1},TUx_{2n+1})]$
$+G[d(Qx_{2n},TUu)+d(Mx_{2n+1},VWx_{2n+1})]$
$+H[d(Mx_{2n},TUx_{2n+1})+d(Px_{2n+1},RSu)]$
$+I[d(Pu,VWx_{2n+1})+d(Qx_{2n+1},TUu)]$
$+J[d(Qx_{2n},TUx_{2n+1})+d(RSu,Mx_{2n+1})]$
$+K[d(Qx_{2n},VWx_{2n+1})+d(TUu,Mx_{2n+1})]$
$+d(TUx_{2n+1},z)+ d(VWx_{2n+1},z)$
taking the limit as $n\to \infty$
d(RSu,z)+(TUu,z)≤Ad(z,z)+Bd(z,z)+Cd(z,z)
$+D[d(Mx_{2n},RSu)+d(Px_{2n+1},TUx_{2n+1})]$
$+E[d(Pu,TUu)+d(Qx_{2n+1},VWx_{2n+1})]$
+F[d(z,RSu)+d(z,z)] +G[d(z,TUu)+d(z,z)]
+H[d(z,z)+d(z,RSu)] +I[d(z,z)+d(z,TUu)]
+J[d(z,z)+d(RSu,z)] +K[d(z,z)+d(TUu,z)]
+d(z, z)+ d(z, z)
d(RSu,z)+d(TUu,z)
≤(D+F+H+J)d(z,RSu)+(E+G+I+K)(z,TUu)
a contradiction. Since,
D+F+H+J<1 and E+G+I+K<1
And hence RSu=Pu=z

Also, RS(X)⊆Q(X) there exist v∈ X such that z = Qv, then by equation (1.2)
d(z,TUv)+d(z,VWv)=d(RSu,TUv) +d(TUu, VWv)
≤Ad(Mu,Pv)+Bd(Pu,Qv)+Cd(Qu,Mv)
+D[d(Mu,z)+d(z,TUv)] +E[d(Pu,TUu)+d(Qv,VWv)]
+F[d(Qu,RSu)+d(z,TUv)] +G[d(Qu,TUu)+d(z,VWv)]
+H[d(Mu,TUv)+d(z,z)] +I[d(z,VWv)+d(z,TUu)]
+J[d(Qu,TUv)+d(z,z)] +K[d(Qu,VWv)+d(TUu,z)]
d(z,TUv)+d(z,VWv)=d(RSu,TUv) +d(TUu, VWv)
≤Ad(Mu,z)+Bd(z,z)+Cd(z,Mv)+D[d(Mu,z)+d(z,TUv)]
+E[d(z,TUu)+d(z,VWv)]+F[d(z,z)+d(Mv,TUv)]
+G[d(z,TUu)+d(Mv,VWv)]+H[d(Mu,TUv)+d(Pv,RSu)]
+I[d(Pu,VWv)+d(Qv,TUu)]+J[d(Qu,TUv)+d(RSu,Mv)]

+K[d(Qu,VWv)+d(TUu,Mv)]
$\leq$Ad(Mx_{2n},z)+Bd(z,z)+Cd(z,z)
+D[d(Mx_{2n},z)+d(z,TUv)] +E[d(z,TUx_{2n})+d(z,VWv)]
+F[d(z,z)+d(z,TUv)] +G[d(z,TUx_{2n})+d(z,VWv)]
+H[d(Mx_{2n},TUv)+d(z,z)] +I[d(z,VWv)+d(z,TUx_{2n})]
+J[d(z,TUv)+d(z,Mx_{2n+1})]+K[d(z,VWv)+d(TUx_{2n},Mx_{2n+1})]
Taking the limit as n→ ∞
$\leq$Ad(z,z)+Bd(z,z)+Cd(z,z)
+D[d(z,z)+d(z,TUv)] +E[d(z,z)+d(z,VWv)]
+F[d(z,z)+d(z,TUv)] +G[d(z,z)+d(z,VWv)]
+H[d(z,TUv)+d(z,z)] +I[d(z,VWv)+d(z,z)]
+J[d(z,TUv)+d(z,z)] +K[d(z,VWv)+d(z,z)]
d(z,TUv)+d(z,VWv)=(D+F+H+J)
d(z,TUv))+(G+I+K)d(z,VWv)
d(RSu,z)+d(TUu,z)
$\leq$(D+F+H+J)d(z,RSu)+(E+G+I+K)(z,TUu)
a contradiction. Since,
D+F+H+J<1 and E+G+I+K<1
And hence TUv=Qv=z
Similarly we can prove that z=Mw=VWw
Now we claim that z is a fixed point of TU, if ≠z, by (1.2) we have
d(z,TUz)+d(TUz,VWz)=d(Rsz,TUz)+d(TUz,VWz)
$\leq$
Ad(VWz,RSz)+Bd(RSz,TUz)+Cd(TUz,VWz)+D[d(z,z)+d(RSz,TUz)]
+E[d(RSz,TUz)+d(TUz,VWz)]+F[d(TUz,RSz)+d(VWZ,TUz)]
+G[d(z,z)+d(z,z)]+H[d(VWz,TUz)+d(z,z)]
+I[d(RSz,VWz)+d(z,z)]+J[d(z,z)+d(RSz,VWz)]
+K[d(TUz,VWz)+d(TUz,VWz)]
$\leq$[A+B+D+E+F]d(RSz,TUz)+[C+A+E+F+H+K+D]d(TUz,VWz)
Therefore = z. hence Qz=z
⇨TUz=Pz=Qz=RSz=VWz=Mz. So z is a common fixed point of P,Q,M,T,U,V,W

By commuting condition conditions of pairs, we have
Tz=T(TUz)=T(UTz)=TUTz
Tz=T(Pz)=P(Tz) and Uz=U(TUz)=(UT)(Uz)=(TU)(Uz)
Uz=U(Pz)=P(Uz), which follows that Tz and Uz are common fixed points of (TU,P)
Then Tz=z=Uz=Pz=TUz
Also, Sz=R(RSz)=R(SRz)=RSRz
Rz=R(Qz)=Q(Rz) and Sz=S(RSz)=(SR)(Sz)=(RS)(Sz)
Sz=S(Qz)=Q(Sz), which follows that Rz and Sz are common fixed points of (RS,Q)
Then Rz=z=Sz=Qz=RSz
Similarly,
Vz=z=Wz=Mz=VWz
Therefore z is a common fixed point of M,P,Q,R,S,T,U,V,W.

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