On the Homogeneous Biquadratic Equation with 5 Unknowns $x^4 - y^4 = 50(z^2 - w^2)R^2$

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Abstract— Five different methods of the non-zero integral solutions of the homogeneous biquadratic Diophantine equation with five unknowns $x^4 - y^4 = 50(z^2 - w^2)R^2$ are determined. Some interesting relations among the special numbers and the solutions are observed.

Key words: Homogeneous equation, Integral solutions, Polygonal numbers, and special number. 2010 Mathematics Subject Classification: 11D09

NOTATIONS USED

$t_{m,n}$ = Polygonal number of rank n with sides m.
s_n = Star number
p_n = Pronic number
G_n = Gnomonic number
W_n = Woodhall number

I. INTRODUCTION

The number theory is queen of Mathematics. In particular, the Diophantine equations have a blend of attracted interesting problems. For an extensive review of variety of problems, one may refer to [3-12]. In this work, we are observed another interesting five different methods of the non-zero integral solutions of the homogeneous biquadratic Diophantine equation with five unknowns $x^4 - y^4 = 50(z^2 - w^2)R^2$. Further, some elegant properties among the special numbers and the solutions are exposed

II. METHOD OF ANALYSIS

The homogeneous biquadratic Diophantine equation with five unknowns to be solved for its non-zero distinct integral solution is

$$x^4 - y^4 = 50(z^2 - w^2)R^2 \quad (1)$$

Consider the transformations

$$x = u + v,\ y = u - v,\ z = 2uv + 1,\ w = 2uv - 1 \quad (2)$$

On substituting (2) in (1), we get

$$u^2 + v^2 = 50R^2 \quad (3)$$

A. Pattern: I

Assume $50 = (7 + i)(7 - i)$\ (4)

and $R = a^2 + b^2 = (a + i b)(a - i b)$ (5)

Using (4) and (5) in (3) and employing the method of Factorization, we get.

$$(u + iv)(u - iv) = (7 + i)(7 - i)(a + ib)^2 (a - ib$$

On equating the positive and negative factors, we have,

$$(u + iv) = (7 + i)(a + ib)^2$$
$$(u + iv) = (7 - i)(a - ib)^2$$

On equating real and imaginary parts, we get

$$u = u(a, b) = 7a^2 - 7b^2 - 2ab$$
$$v = v(a, b) = a^2 - b^2 + 14ab$$

On substituting u and v in (2) we get the values of x, y, z and w. The non-zero distinct integrals values of x, y, z, w and R satisfying (1) are given by

$$x = x(a, b) = 8a^2 - 8b^2 + 12ab$$
$$y = y(a, b) = 6a^2 - 6b^2 - 16ab$$
$$z = z(a, b) = 2(7a^4 + 7b^4 - 42a^2 b^2 + 96a^3b - 96ab^3) + 1$$
$$w = w(a, b) = 2(7a^4 + 7b^4 - 42a^2 b^2 + 96a^3b - 96ab^3) - 1$$
$$R = R(a, b) = a^2 + b^2$$

B. Properties:

1) $x(a, 1) + y(a, 1) - S_a - P_a - 4t_4,\ a \equiv 0 \pmod 5$
2) $R(1, 2a) - 4t_{4,a} - 1 = 0$
3) $z(2a, 1) - 3[x(a, 1) + y(a, 1)] - 38\, t_{4a} + G_{6a} + \equiv 0 \pmod 2$
4) $R(a+1, a+1) - 2t_{4,a} - 2G_a \equiv 0 \pmod 3$
5) $x(2, b) + y(2, b) + 14t_{4,b} + 4G_b \equiv 0 \pmod{57}$

1) *Pattern: II*

Also 50 can be chosen in equation (3) as

$$50 = (1 + 7i)(1 - 7i) \quad (6)$$

Using (5) and (6) in equation (3) it is written in factorizable from as

$$(u + iv)(u - iv) = (1 + 7i)(1 - 7i)(a + ib)^2 (a - ib)^2$$

On equating the positive and negative factors,

We get,

$$(u + iv) = (1 + 7i)(a + ib)^2$$
$$(u - iv) = (1 - 7i)(a - ib)^2$$

On equating real and imaginary parts, we have

$$u = u(a, b) = a^2 - b^2 - 14ab$$
$$v = v(a, b) = 7a^2 - 7b^2 + 2ab$$

Substituting the values of u and v in (2), the non-zero distinct values of x, y, z,w and R satisfying (1) are given by

$$x = x(a, b) = 8a^2 - 8b^2 - 12ab$$
$$y = y(a, b) = -6a^2 + 6b^2 - 16ab$$
$$z = z(a, b) = 2(7a^4 + 7b^4 - 42a^2 b^2 - 96a^3b + 96ab^3) + 1$$
$$w = w(a, b) = 2(7a^4 + 7b^4 - 42a^2 b^2 - 96a^3b - 96ab^3) - 1$$
$$R = R(a, b) = a^2 + b^2$$

2) *Properties:*

- $x(a, 1) + R(a, 1) - 9t_{4,a} - G_{6a} \equiv 0 \pmod 2$
- $y(1, b) + (R(1, 2b))^2 - 4w_4 + 4t_{4,b} - G_{8b} \equiv 0 \pmod 3$
- $R(3a, 1) + 8a - 9t_{4,a} - G_{4a} \equiv 0 \pmod 2$
- $R(2a, 2a) - 8t_{4,a} = 0$
- $x(a, a + 1) + y(a, a + 1) + 28t_{4,a} - 2 = 0$

C. Pattern: III

Rewrite (3) as

$$1 * u^2 = 50R^2 - v^2 \quad (7)$$

Assume $u = 50a^2 - b^2 = (\sqrt{50}\,a + b)(\sqrt{50}\,a - b)$ (8)

Write 1 as $1 = (\sqrt{50} + 7)(\sqrt{50} - 7)$ (9)

Using (8) and (9) in (7) it is written in factorizable form as,

$$(\sqrt{50} + 7)(\sqrt{50} - 7)(\sqrt{50}\,a + b)^2 (\sqrt{50}\,a - b)^2$$

$$= (\sqrt{50}R + v)(\sqrt{50}R - v) \quad (10)$$

On equating the rational and irrational parts, we get

$$(\sqrt{50} + 7)(\sqrt{50}a + b)^2 = (\sqrt{50}R + v)$$
$$(\sqrt{50} - 7)(\sqrt{50}a - b)^2 = (\sqrt{50}R - v)$$

On equation the real and imaginary parts, we get

$$R = R(a, b) = 50a^2 + b^2 + 14ab$$
$$V = v(a, b) = 350a^2 + 7b^2 + 100ab$$

Substituting the values of u and v in (2), the non – zero distinct integral values of x, y, z, R and w satisfying (1) are given by

$$x = x(a, b) = 400a^2 + 6b^2 + 100ab$$
$$y = y(a, b) = -300a^2 - 8b^2 - 100ab$$
$$z = z(a, b) = 2(17500a^4 - 7b^4 - 100ab^3 + 5000a^3b) + 1$$
$$w = w(a, b) = 2(17500a^4 - 7b^4 - 100ab^3 + 5000a^3b) - 1$$
$$R = R(a, b) = 50a^2 + b^2 + 14ab$$

1) Properties:

- $R(n+1, 1) - (t_{16,n} - 42t_{4,n} - 53G_n \equiv 0 \pmod 5)$
- $x(n+1, n+2) + y(n+3, 1) - 206t_{4,n} - 388G_n \equiv 0 \pmod{2383}$
- $R(1, 2n) - 4t_{4,n} - 14G_n \equiv 0 \pmod 3$
- $x(2n, 2n) - 2024\, t_{4,n} = 0$
- $R(3, 3n) - 9t_{4,n} - 63\, G_n \equiv o\ (mod11)$

D. Pattern: 4

Rewrite (3) as

$$1 * v^2 = 50R^2 - u^2 \quad (11)$$

Write 1 as $1 = \dfrac{(\sqrt{50}-1)(\sqrt{50}+1)}{49}$ (12)

Assume $v = 50a^2 - b^2 = (\sqrt{50}a - b)(\sqrt{50}a + b)$ (13)

Using (12) and (13) in (11), it is written in factorizable form as,

$$\frac{(\sqrt{50}-1)(\sqrt{50}+1)}{49}(\sqrt{50}a - b)^2(\sqrt{50}a + b)^2 =$$
$$(\sqrt{50}R - u)(\sqrt{50}R + u) \quad (14)$$

On equating the rational and irrational factors we get,

$$R = R(a, b) = \frac{1}{7}(50a^2 + b^2 + 2ab)$$
$$u = u(a, b) = \frac{1}{7}(50a^2 + b^2 + 100ab) \quad (15)$$

Replacing ‘a’ by 7A and ‘b’ by 7B in the above equations (13) and (15), we get

$$R = R(A, B) = 350A^2 + 7B^2 + 14AB$$
$$u = u(A, B) = 350A^2 + 7B^2 + 700AB$$
$$v = v(A, B) = 2450A^2 - 49B^2$$

On substituting the values of u and v in (2), the non –zero distinct integrals values of x, y, z, w and R satisfying (1) are given by

$$x = x(A, B) = 2800A^2 - 42B^2 + 700AB$$
$$y = y(A, B) = -2100A^2 + 56B^2 + 700AB$$
$$z = z(A, B) = 2(257500A^4 - 343B^4 + 33500A^3B + 686AB^3) + 1$$
$$w = w(A, B) = 2(257500A^4 - 343B^4 + 33500A^3B - 686AB^3) - 1$$
$$R = R(A, B) = 350A^2 + 7B^2 + 14AB$$

1) Properties:

- $R(A, 1) - x(A, 1) + 2450t_{4,A} - 352G_A \equiv 0 \pmod 2$
- $R(1, 2A) - y(1, 2A) - 28P_A - t_{450,A} - t_{2358,A} + 1178t_{4,A} \equiv 0 \pmod 5$
- $x(1, 4A) + t_{1346,A} - t_{4262,A} + 2130\, t_{4,A} \equiv 0 \pmod 2$
- $y(1, 3A) - 504\, t_{4,A} - t_{4204,A} + 2101t_{4,A} \equiv 0 \pmod 5$
- $x(2A, 2A) - 13832t_{4,A} = 0$

E. Pattern: 5

Write (3) as

$$(u + R)(u - R) = (7R + v)(7R - v) \quad (16)$$

Which is expressed in the form of ratio as

$$\frac{u+R}{7R+v} = \frac{7R-v}{u-R} = \frac{A}{B}, B \neq 0 \quad (17)$$

This is equivalent to the following two equations

$$-uA + R(7B + A) - VB = 0$$
$$uB + R(B - 7A) - VA = 0$$

On solving the above equations by the method of cross multiplication we get,

$$u = u(A, B) = -A^2 - B^2$$
$$R = R(A, B) = A^2 + B^2$$
$$v = v(A, B) = -7A^2 + 7B^2 + 2AB$$

Substituting the values of u and v in (2), the non – zero distinct integral values of x, y, z, w and R satisfying (1) are given by,

$$x = x(A, B) = -8A^2 + 6B^2 + 2AB$$
$$y = y(A, B) = 6A^2 - 8B^2 + 2AB$$
$$z = z(A, B) = 2[7A^4 - 7B^4 - 2A^3B - 2AB^3] + 1$$
$$w = w(A, B) = 2[7A^4 - 7B^4 - 2A^3B - 2AB^3] - 1$$
$$R = R(A, B) = A^2 + B^2$$

1) Properties:

- $x(A, 1) + R(A, 1) + 4t_{4,A} + t_{8,A} \equiv 0 \pmod 7$
- $y(A, A+1) + 2t_{4,A} + 8G_A - 2P_n \equiv 0 \pmod 7$
- $R(2, 2A) - 4t_{4,A} \equiv 0 \pmod 2$
- $R(A+1, A+2) - t_{6,A} - 3G_A - P_n + t_{4,A} - 6 = 0$
- $R(3A, 3A) - 18t_{4,A} = 0$

III. Conclusion

It is worth to note that in (2), the transformations for z and w maybe considered as z= 2u + v and w =2u – v. For this case, the values of x, y and R are the same as above where as the values of z and w changes for every pattern. To conclude one may consider biquadratic equations with multivariables ($\geq$5) and search for their non-zero distinct integer solutions along with their corresponding properties.

References

[1] Dickson, L.E., History of theory of numbers,Vol.11, Chelsea publishing company, New –York (1952)
[2] Mordell, L.J., Diophantine equation, Academic press, London (1969) Journal of Science and Research, Vol (3) Issue 12, 20-22 (December-14)
[3] Jayakumar. P, Sangeetha, K “Lattice points on the cone $x^2+9y^2=50z^2$” International Journal of Science and Research, Vol (3), Issue 12, 20-22 (December -2014)
[4] Jayakumar P, Kanaga Dhurga, C,” On Quadratic Diophantine equation $x^2 + 16y^2 = 20z^2$” Galois J. Maths, 1(1) (2014), 17-23
[5] Jayakumar. P, Kanaga Dhurga. C, “Lattice points on the cone $x^2 + 9y^2 = 50z^2$” Diophantus J. Math, 3(2) (2014), 61-71
[6] Jayakumar. P, Prabha. S “On Ternary Quadratic Diophantine equation $x^2+15y^2=14z^2$” Archimedes J. Math., 4(3) (2014), 159-164
[7] Jayakumar, P, Meena, J “Integral solutions of the Ternary Quadratic Diophantine equation: $x^2 + 7y^2 = 16z^2$” International Journal of Science and Technology, Vol.4, Issue 4, 1-4, Dec 2014

[8] Jayakumar. P, Shankarakalidoss, G “Lattice points on Homogenous cone $x^2 + 9y^2 = 50z^2$” International journal of Science and Research, Vol (4), Issue 1, 2053-2055, January -2015.

[9] Jayakumar. P, Shankarakalidoss. G “Integral points on the Homogenous cone $x^2 + y^2 = 10z^2$” International Journal for Scienctific Research and Development, Vol (2), Issue 11, 234-235, January -2015

[10] Jayakumar. P, Sangeetha. K, “Integral solution of the Homogeneous Biquadratic Diophantine equation with six unknowns: $(x^3 - y^3) z = (W^2 - P^2)R^4$ “International Journal of Science and Research, Vol(3), Issue 12, 1021-1023 (December-2014)

[11] Jayakumar. P, Shankarakalidoss. G “Integral Solution of the Homogeneous Biquadratic Diophantine Equations with five Unknowns International Journal of Science and Research, Vol (4), Issue 3, 40 - 42, March - 2015

$$(X^2 - Y^2)(3X^2 + 3Y^2 - 2XY) = 12(Z^2 - W^2)T^2$$":

[12] Jayakumar. P, Shankarakalidoss. G “Integral Solution of the Homogeneous Biquadratic Diophantine Equations with Five Unknowns ”International Journal for Scientific Research and Development, Vol (2), Issue 12 , 722-724, 2015

$$(X^2 - Y^2)(2X^2 + 2Y^2 - 3XY) = 11(Z^2 - W^2)T^2$$

