

# STABILITY ANALYSIS OF THE SYSTEM BASED ON INVERTED PENDULUM WITH DOUBLE-POLE ARRANGEMENT USING LQR CONTROLLER

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**Abstract---** The double-pole inverted pendulum system on a cart is highly nonlinear and unstable system which invoke the researchers to find the ways of stability with different controllers. In this paper we have demonstrated the mathematical modeling and by applying the controllability principle we found this system is inherently unstable and then we applied a modern LQR approach with different Q and R matrices weightage to check the stability.

## I. INTRODUCTION

One of the most exciting research projects is to study humanoid robot. Hence it is important to understand the mathematical modeling and theoretical background of humanoid robot. In standing it has become common to consider your body as an (single/double/triple) inverted pendulum pivoted at ankles. Moreover up ride of human shoulder is also considered as motion of (single/double/triple) inverted pendulum.

In the past few years, the analysis and control of fully actuated robot manipulators has been extensively studied. Many control strategies based on passivity, Liapunov theory, feedback linearization, adaptability, robustness, learning, etc. An inverted pendulum system is a typically nonlinear, redundancy, uncertainty, strong coupling and natural characteristics of instabilities. All these features make it the ideal model of advanced control theory and typical research platform of test control results.

There are a number of different kinds of the inverted pendulum systems presenting a variety of control challenges. The most common types are the single inverted pendulum on a cart, the double inverted pendulum on a cart, the double inverted without cart (classical double inverted pendulum).

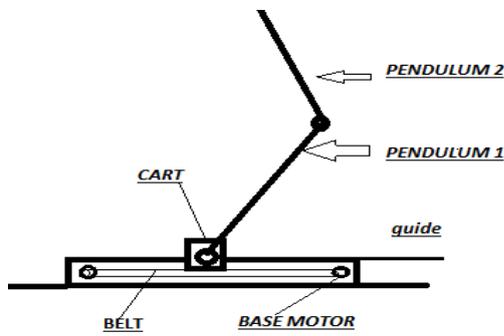


Fig. 1: Set Up Of Double-Pole Inverted Pendulum

Balancing an inverted pendulum system is analogous to balance a broom stick on your palm or index finger. In the case of broomstick to keep the broomstick up right you have to constantly adjust the position of your hand.

While in case of cart-pole system cart does basically the same thing as balancing equipment.[3]

Your hand can move in any direction up, down, sideways (left, right) etc. while cart pole system can move only in one direction (forward, backward). Force must be applied in proper way to keep the system stable.

To achieve this goal a proper control theory is required.[3] So basically pole attached to the cart (inverted pendulum) is an unstable system like a broomstick. And double pole inverted pendulum is another (2<sup>nd</sup>) pole attached to the other end 1<sup>st</sup> pole.

There are various applications where the concept of inverted pendulum can be applicable to stabilize systems like Ships, Underwater vehicles, helicopters, Aircraft, satellites, space platforms Mobile robots, Flexible joint robots, Hyper-redundant and snake-like manipulators, Walking robots, capsule robots, and hybrid machines, Military application (missiles and rockets) ,Self-balancing robots.[1]

## II. MATHEMATICAL MODELLING

Double inverted pendulum consist of cart, two pendulums, and guide as a motor. There are two techniques used obtained mathematical modelling.

1 Newton's law of motion.ig 2

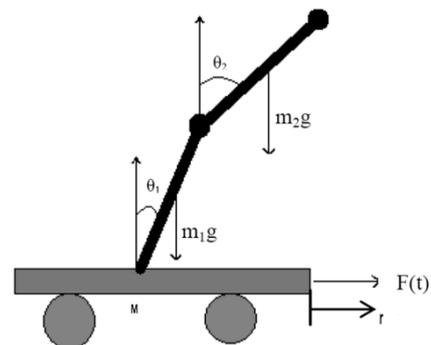


Fig. 2: Schematic Diagram Of Double Pole Inverted Pendulum.

For mathematical modelling system is divided into three parts

- (1) Cart.
- (2) 1<sup>st</sup> pendulum.
- (3) 2<sup>nd</sup> pendulum.

Three types of energies are considered on each part.

A. For the Lower Pendulum:

Kinetic energy:

$$K_1 = \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}m_1 \left\{ \left[ \frac{d}{dt}(l_1 \sin \theta_1) \right]^2 + \left[ \frac{d}{dt}(l_1 \cos \theta_1) \right]^2 \right\} \dots (1)$$

Potential Energy:

$$P_1 = m_1 g l_1 \cos \theta_1 \dots (2)$$

Dissipation Energy:

$$D_1 = \frac{1}{2}c_1\dot{\theta}_1^2 \dots (3)$$

B. For the upper pendulum:

Kinetic Energy:

$$K_2 = \frac{1}{2}J_2\dot{\theta}_2^2 + \frac{1}{2}m_2 \left\{ \left[ \frac{d}{dt}(L \sin \theta_1 + l_2 \sin \theta_2) \right]^2 + \left[ \frac{d}{dt}(L \cos \theta_1 + l_2 \cos \theta_2) \right]^2 \right\} \dots \dots (4)$$

Potential Energy:

$$P_2 = m_2 g (L \cos \theta_1 + l_2 \cos \theta_2) \dots (5)$$

Dissipation Energy :

$$D_2 = \frac{1}{2}c_1(\dot{\theta}_1^2 + \dot{\theta}_2^2) \dots (6)$$

C. For the cart:

Kinetic Energy:

$$K_3 = \frac{1}{2}M\dot{r}^2 \dots (7)$$

Potential Energy:

$$P_3 = 0 \dots (8)$$

Dissipation Energy:

$$D_3 = \frac{1}{2}F\dot{r}^2 \dots (9)$$

where r-is the distance of the cart from the reference position,

$\theta_1$  and  $\theta_2$  are the angles of the lower pendulum and the upper pendulum from the vertical axis

$m_i$  -denote the mass,

$l_i$  - the distance between the center of the hinge its center of gravity,

$J_i$  -is moment of inertia

$c_i$  -is the friction coefficient for rotation between the pendulum and hinge.

L is the length (between the hinges) of the lower pendulum.

Then the following relations exist (summing the kinetic, potential and dissipation energies)

$$\frac{d}{dt} \left( \frac{\partial K}{\partial \dot{r}} \right) - \frac{\partial K}{\partial r} + \frac{\partial P}{\partial r} + \frac{\partial D}{\partial r} = ae \dots \dots (11)$$

$$\frac{d}{dt} \left( \frac{\partial K}{\partial \dot{\theta}_i} \right) - \frac{\partial K}{\partial \theta_i} + \frac{\partial P}{\partial \theta_i} + \frac{\partial D}{\partial \theta_i} = 0, \dots (12)$$

Where a is the gain of overall cart driving system and e is the input to Voltage to the amplifier satisfying  $e_0 \geq |e|$ ,

$$\bar{x} = \begin{bmatrix} r \\ \theta_1 \\ \theta_2 \end{bmatrix}, u e_0 = e.$$

Equation (12) can be written as

$$K_1 \bar{x}'' = K_2 \bar{x}' + K_3 + K_4 u \dots (13)$$

Where

$$K_1 = \begin{bmatrix} m_1 + m_2 + M & (m_1 l_1 + m_2 L) \cos \theta_1 & m_2 l_2 \cos \theta_2 \\ (m_1 l_1 + m_2 L) \cos \theta_1 & J_1 + m_1 l_1^2 + m_2 L^2 & m_2 l_2 L \cos(\theta_1 - \theta_2) \\ m_2 l_2 \cos \theta_2 & m_2 l_2 L \cos(\theta_1 - \theta_2) & J_2 + m_2 l_2^2 \end{bmatrix} \dots (14)$$

$$K_2 = \begin{bmatrix} -F & (m_1 l_1 + m_2 L) \theta_1' \sin \theta_1 & m_2 l_2 \theta_2' \sin \theta_2 \\ 0 & -c_1 - c_2 & -m_2 l_2 L \theta_2' L \sin(\theta_1 - \theta_2) + c_2 \\ 0 & -m_2 l_2 L \theta_2' L \sin(\theta_1 - \theta_2) + c_2 & c_2 \end{bmatrix} \dots (15)$$

$$K_3 = \begin{bmatrix} 0 \\ (m_1 l_1 + m_2 L) g \sin \theta_1 \\ m_2 l_2 g \sin \theta_2 \end{bmatrix} \dots (16)$$

$$K_4 = \begin{bmatrix} e_0 a \\ 0 \\ 0 \end{bmatrix} \dots (17)$$

Now defining

$$x = \begin{bmatrix} \bar{x} \\ \bar{x}' \end{bmatrix}$$

Then

$$\bar{x}' = \begin{bmatrix} \bar{x}' \\ K_1^{-1}(K_2 \bar{x}' + K_3 + K_4 u) \end{bmatrix}$$

Gives the mathematical model for the double inverted pendulum system,

Where

$$\dot{x} = x'$$

The system is nonlinear, and containing complex terms.

For making the system linear taking  $x=0$  and linear model is obtained.

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

Where u is the input and y is the output and the vector  $x \in \mathbb{R}^n$  is called the state vector.

$$A, B, C \text{ are constant matrix where } A = \begin{bmatrix} 0 & I_3 \\ L_1^{-1} K_3' & L_1^{-1} L_2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ L_1^{-1} K_4' \end{bmatrix}$$

$$K_3' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & (m_1 l_1 + m_2 L) g & 0 \\ 0 & 0 & m_2 L g \end{bmatrix}$$

$$K_4 = \begin{bmatrix} e_0 a \\ 0 \\ 0 \end{bmatrix}$$

where  $\theta_1 = \theta_2 = 0$

$$L_2 = \begin{bmatrix} -F & 0 & 0 \\ 0 & -c_1 - c_2 & c_2 \\ 0 & c_2 & c_2 \end{bmatrix} = K_2$$

where  $\theta_1 = \theta_2 = \theta_1' = \theta_2' = 0$

For the analysis and controllability and observability of a control system the following equivalent system is used

$$\dot{x} = Ax + Bu$$

$$y = Cx = [I_3 \ 0]x$$

where  $I_3$  is the  $3 \times 3$  identity matrix, the vector x is the state, the 3-vector y is the output and the scalar u is the input of the system.

$$[A : B] = \begin{bmatrix} 0 & I_3 & 0 \\ A_{21} & A_{22} & B_2 \end{bmatrix}$$

Where

$$A_{21} = L_1^{-1} K_3^{-1},$$

$$A_{22} = L_1^{-1}L_2,$$

$$B_2 = L_1^{-1}K_4$$

Thus a mathematical model of the double inverted pendulum has been derived.

### III. MATLAB PROGRAMMING OF MATHEMATICAL MODELLING

Taking the parameters which are given in following table and Ref.[1] for MATLAB programming

Table 1. Parameters for the double inverted pendulum system.[1]

Parameters	Values
$m_1$	0.4 kg
$m_2$	0.2 kg
M	0.42 kg
$l_1$	0.43 m
$l_2$	0.2 m
L	0.2 m
$J_1$	0.048 kgm <sup>2</sup>
$J_2$	0.0054 kgm <sup>2</sup>
$c_1$	0.0054 kgm <sup>2</sup> s <sup>-1</sup>
$c_2$	0.00368 kgm <sup>2</sup> s <sup>-1</sup>
F	1.87 kgs <sup>-1</sup>
A	46.7 NV <sup>-1</sup>
G	9.8 ms <sup>-2</sup>

By substituting all parameters in the linearized model of the system we have values of the system matrix A-B.

$$A = \begin{bmatrix} 0_{3 \times 3} & I_{3 \times 3} \\ A_{21} & A_{22} \end{bmatrix}$$

Where

$$A_{21} = \begin{bmatrix} 0 & -5.2901 & -0.11 \\ 0 & 32.0079 & -4.54 \\ 0 & -57.1503 & 30.9931 \end{bmatrix}$$

$$A_{22} = \begin{bmatrix} -2.8504 & 0.0190 & -0.0069 \\ 5.1380 & -0.1662 & 0.0927 \\ -4.8934 & 0.5116 & -0.3804 \end{bmatrix}$$

### IV. STABILITY ANALYSIS OF THE UNCOMPENSATED SYSTEM

After substituting all parameters and find out system matrixes evaluating the Eigen values of the system matrix A are [0 6.53 3.6786 -7.4320 -4.5163 -1.6637]

We have two positive eigenvalues and one zero eigenvalue. Therefore the system is unstable. Now we will check the controllability and observability for the system.

#### A. Observability:

A linear system is completely observable if and only if the rank of the following matrix define is equal to  $n$

$$O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

#### B. Controllability:

A linear system is completely controllable if and only the rank of the controllability matrix defined as follow is equal to  $n$ ;

$$C_o = (B \ BA \ \dots \ \dots \ BA^{n-1})$$

Because of the rank of both the matrix is equal 6 for the double pendulum system. Hence we have that the double inverted pendulum system is controllable and observable. Therefore we are able to stabilize the system and design a LQR controller.

### LQR CONTROLLER DESIGN

Linear Quadratic Regulator (LQR) system which deals with state regulation, output regulation, and tracking. we are interested in the design of optimal linear systems with quadratic performance indices [12].We will now consider the optimal regulator problem that, given the system equation.

$$\dot{x}(t) = A x(t) + B u(t)$$

$$Y(t) = C x(t) + D u(t) \dots(17)$$

Determine the matrix K of the optimal control vector.  $u$

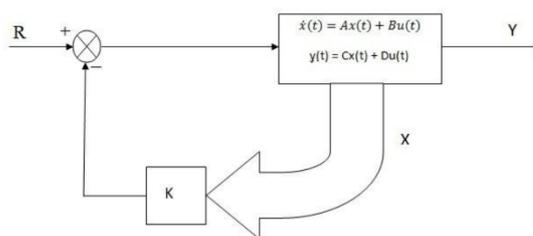


Fig. 3: feedback gain of LQR controller

$$u(t) = -Kx(t) \quad (18)$$

So as to minimize the performance index

$$J = \int_0^{\infty} (x^T(t)Qx(t) + U^T Ru(t))dt \dots(19)$$

Where Q is a positive-semi definite and R is a positive-definite matrix. The matrices Q and R will determine the relative error. Here the elements of the matrix K are determined so as to minimize the performance index. Then

$u(t) = -Kx(t) = -R^{-1}B^T P x(t)$  is optimal for any initial  $x(0)$  state.

Using the LQR method, the effect of optimal control depends on the selection of weighting matrices Q and R, If Q and R carefully not chosen, it make the solution cannot meet the actual system performance requirements. In general, Q and R are occupied the diagonal matrix, the current approach for selecting weighting matrices Q and R is simulation of trial, after finding a suitable Q and R, it allows the use of computers to find the optimal gain matrix K easily. According to the state equation which we had developed according to straight line of the inverted pendulum system,

Where P (t) is the solution of Riccati equation, K is the linear optimal feedback matrix. Now we only need to solve the Riccati equation.  $A^T P + PA - PBR^{-1}B^T P + Q = 0 \dots$  (20)

Where Q and R selected as  $Q = \text{diag}[1 \ 1 \ 1 \ 1 \ 1 \ 1]$  and  $R = 1$ . Therefore,  $K = -R^{-1}B^T P = [1.0000 \ -46.2276 \ 71.4712 \ 1.5489 \ 5.8220 \ 13.4760]$

V. SIMULATIONS AND RESULTS

We have taken various cases based on variation of values of Q&R simulation. Response of LQR controller on the system and step and impulse response also shown

Case(1):

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

R=0.1

$$K = [3.1623 \ -192.7642 \ 348.5471 \ 12.6381 \ 30.6808 \ 73.9067]$$

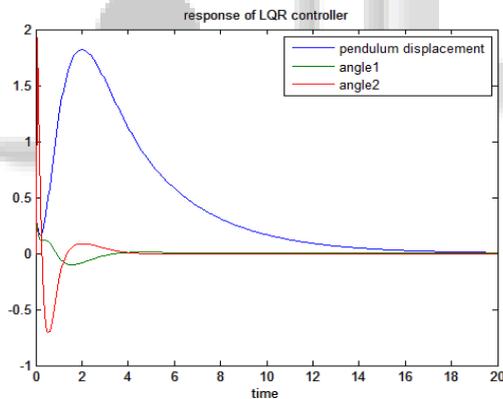


Fig. 4: LQR response.

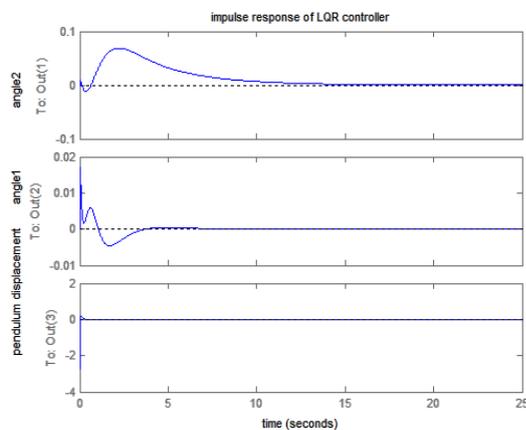


Fig. 5: impulse response after giving LQR controller.

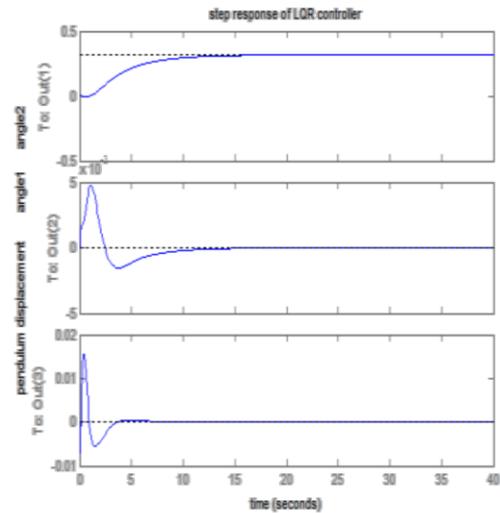


Fig. 6: Step response after giving LQR.

VI. CONCLUSION

Here modelling of DIP shows that the system is unstable when uncompensated. Results of applying LQR controller method shows that the system can be stabilized smoothly. In particularly LQR system If value of Q is large means that, to keep J small, the state x (t) must be smaller. On the other hand if you select R large means that the control input u (t) must be smaller to keep J small. This means that larger values of Q generally result in the poles of the closed-loop system matrix being further left in the s-plane so that the state decays faster to zero. On the other hand, larger R means that less control effort is used, so that the poles are generally slower, resulting in larger values of the state x(t).after applying STEP and IMPULSE signals to the system response of the system is stable.

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