Comparison of Interpolation Filter Structures
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Abstract—The design and implementation of interpolation filters has been a subject of intense research for the last few years. Different interpolation filter designs have been proposed using different design algorithms and implementation structures. In this paper, a review of different designs of interpolation filters and the implementation structures has been presented.

Key words: Direct Form, Finite Impulse Response Filter, Interpolation Filter and Polyphase

I. INTRODUCTION

Interpolation (Bellanger et al., 1974; Crochiere and Rabiner, 1981; Schaffer and Rabiner, 1973) is basically up-sampling and filtering the signal. If \( L \) is the factor of interpolation then it can be implemented in two steps. Up-sampling of the original signal by inserting \( L-1 \) zero-valued samples between two consecutive samples and Removal of the \( L-1 \) images from the spectrum of the up-sampled signal using an anti-imaging or interpolation filter. The up-sampling by an integer factor \( L \) is performed by inserting \( L-1 \) zeros between two consecutive samples.

Fig. 1: Block Diagram of an Interpolator

Applying the up-sampling operation to the discrete signal \( x(n) \), produces the up-sampled signal \( u(m) \) where

\[
u(m) = \begin{cases} x(m/L), & m = 0, \pm L, \pm 2L, \ldots \ldots \\ 0, & \text{otherwise} \end{cases}
\]

The up-sampling operation increases the sampling rate \( F' \) of the original signal \( x(n) \). The sampling frequency \( F' \) of the signal \( u(m) \) is \( L \) times larger than the sampling rate of the original signal, i.e. \( F' = LF \). The up-sampling is sometimes called the sequence expansion. Fig. 2 shows the time domain as well as the spectrum of up-sampling operation by a factor \( L = 3 \). The signal \( x(n) \) gets upsampled by filling-up \( L-1 \) zero valued samples between each of its two consecutive samples (Proakis and Manolakis, 2006).

To obtain the relationship between \( x(n) \) and \( u(m) \) in frequency domain, we have to find the \( z \)-transform of \( u(m) \) on the unit circle:

\[
U(z) = \sum_{m=-\infty}^{\infty} u(m)z^{-m} = \sum_{n=-\infty}^{\infty} x(m)z^{-mL}.
\]

As shown in Fig. 2, the spectrum of \( u(m) \) not only contains the required signal from \( -\pi / L \) to \( \pi / L \), but also the images of the signal \( x(n) \) centered at frequencies between the original Nyquist frequency and the higher interpolated Nyquist frequency i.e. \( \pm 2\pi / L, \pm 4\pi / L, \ldots \ldots \). To mitigate the effect of imaging, an anti-imaging low pass interpolation filter is used to remove images from the spectrum of the upsampled signal. Removal of images from the spectrum of the signal causes the interpolation of the sample values in time domain. The zero-valued samples in the up-sampled signal \( u(m) \) are “filled-in” with the interpolated values to obtain the interpolated signal \( y(m) \). The characteristics of interpolation filter must approximate the ideal characteristics of a low pass filter given by (Crochiere and Rabiner, 1981)

\[
H(e^{j\omega}) = \begin{cases} C, & \left| \omega \right| \leq \pi / L \\ 0, & \text{otherwise} \end{cases}
\]

The gain of the filter \( C \) must be equal to interpolation factor \( L \) in passband. The output interpolated signal in frequency domain is given by (Crochiere and Rabiner, 1981)

\[
Y(e^{j\omega}) = \begin{cases} CX(e^{j\omega}), & \left| \omega \right| \leq \pi / L \\ 0, & \text{otherwise} \end{cases}
\]
In time domain the interpolated signal \( y(m) \) is given by

\[
y(m) = \sum_{k=-\infty}^{\infty} h(m-k)u(k)
\]  
(8)

Where \( h(m) \) is the unit sample response of \( H(e^{j\omega}) \).

Now put equation (1) in (8):

\[
y(m) = \sum_{k=-\infty}^{\infty} h(m-k)x(k/L)
\]  
(9)

\[
y(m) = \sum_{k=-\infty}^{\infty} h(m-kL)x(k)
\]  
(10)

After change of variables

\[ k = \frac{m-n}{L} \]  
(11)

Where \( \left\lfloor \frac{m}{L} \right\rfloor \) denotes the largest integer contained in \( \frac{m}{L} \). Substituting equation (11) in (10):

\[
y(m) = \sum_{n=-\infty}^{\infty} h\left(n-L\left\lfloor \frac{m}{L} \right\rfloor \right)x\left(\frac{m-n}{L}\right)
\]  
(12)

Using identity

\[ m-L\left\lfloor \frac{m}{L} \right\rfloor = m \mod L \]

i.e. \( m \mod L \), equation (12) becomes,

\[
y(m) = \sum_{n=-\infty}^{\infty} h(nL+m \mod L)x\left(\frac{m}{L}\right)
\]  
(13)

Hence \( y(m) \) is represented in terms of input signal \( x(n) \) and the interpolation filter coefficients \( h(m) \). It has been shown from equation (13) that the output \( y(m) \) is obtained by passing the input sequence \( x(n) \) through a time-variant filter with impulse response \( g(n,m) \).

\[
g(n,m) = h(nL+m \mod L), \quad \text{for all } n \text{ and all } m
\]  
(14)

From the above discussion, it has been found that the interpolation filter plays a very important role in the interpolation process. The output of the interpolator depends upon the interpolation filter coefficients and in turn the design of interpolation filters. The filter coefficients decide the response of the filter in passband, transition band as well as in stopband. In the next section, different interpolation filter structures have been discussed in detail.

II. DIFFERENT INTERPOLATION FILTER STRUCTURES

The role of filtering in interpolation process is to remove the images from the spectrum of the upsampled signal i.e. to band limit the spectrum of the signal to the prescribed bandwidth in accordance with the actual sampling rate. Since an ideal frequency response cannot be achieved, hence the performance of the system for interpolation is mainly determined by filter characteristics. The specific role of a digital filter in interpolation process is the high-performance filtering with the lowest possible complexity. The interpolation filter (as shown in Fig. 1) can be implemented using different filter structures. The direct implementation is inefficient because the filtering has to be performed on the side of higher sampling rate i.e. up-sampling precedes the filtering operation. Hence, the goal is to construct a filter implementation structure providing the arithmetic operations to be performed at the lower sampling rate and to embed the up-sampling operation into the filter structure. In this way, the overall workload on the system can be decreased by a factor of \( L \). In this section we will discuss about the different implementation structures of interpolators.

III. DIRECT FORM STRUCTURE OF FIR INTERPOLATOR

The direct form structure (Crochiere and Rabiner, 1981; Proakis and Manolakis, 2006; Turek, 2004) for the interpolation filter can be derived from its block diagram (shown in Fig. 1). The signal \( x(n) \) is up-sampled by \( L \) and then filtered by the anti-imaging FIR filter. The input signal to the filter is \( u(m) \) which is the result of the up-sampling operation performed on the input signal \( x(n) \). From equation (5), we see that every \( L \)th input sample to the filter has a non-zero value. Therefore in the conventional FIR filter structure as shown in Fig. 3, \( L-1 \) out of \( L \) input samples are multiplied with the filter coefficients without contributing to the values of the output samples.

![Fig. 3: Conventional Direct Form Structure of FIR Interpolator](image)

Since there is no need to multiply the filter coefficients by the zero-valued samples, we can perform multiplication operations at the sampling rate of the input signal, and then up-sampled by \( L \). The structure has been shown in Fig. 4 where the up-sampling operation is embedded into the filter structure. The up-sampled samples come to the adders and thus arrive to the chain of adders and delays. This implementation structure reduces the number of multiplications per output sample from \( N \) (the filter length) to \( N/L \).
Hence it is efficient to introduce the up-sampling operation into the filter structure so that the multiplication operations must be calculated before the sampling up-conversion.

IV. POLYPHASE STRUCTURE OF FIR INTERPOLATOR

Polyphase structure (Bellanger et al., 1976; Crochiere and Rabiner, 1981; Emami, 1999) of FIR filter is an efficient structure to implement the interpolation filters. This method is used to realize a higher-order FIR interpolation filter in parallel structure and is based on the polyphase decomposition of the filter transfer function. The transfer function $H(z)$ of the filter is to be decomposed into $M$ lower order transfer functions, called the polyphase components, which are then added together to get the original overall transfer function. First let us decompose $H(z)$ into two polyphase components, and then we will generalize the transfer function equation in terms of polyphase decomposition (Bellanger et al., 1976; Salivahanan, et al., 2000).

\[
H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \ldots + h(N-2)z^{-(N-2)} + h(N-1)z^{-(N-1)} \quad (15)
\]

Now decompose this transfer function into two polyphase components with first term from the even-indexed coefficients and the second term from the odd-indexed coefficients.

Assuming $N$ is an odd number,

\[
H(z) = h(0) + h(2)z^{-2} + \ldots + h(N-2)z^{-(N-2)} + h(N-1)z^{-(N-1)} \quad (16)
\]

\[
H(z) = h(0) + h(1)z^{-1} + \ldots + h(N-3)z^{-(N-3)} + h(N-2)z^{-(N-2)} \quad (17)
\]

or

\[
P_2(z) = h(0) + h(2)z^{-2} + \ldots + h(N-3)z^{-(N-3)} \quad (18)
\]

\[
P_1(z) = h(0) + h(1)z^{-1} + \ldots + h(N-3)z^{-(N-3)} \quad (19)
\]

$P_1(z)$ and $P_2(z)$ are the even and odd-indexed coefficients of $H(z)$.

Hence $H(z)$ can be expressed in terms of $P_1(z)$ and $P_2(z)$ as

\[
H(z) = P_1(z^2) + z^{-1}P_2(z^2) \quad (20)
\]

Now for general case, $H(z)$ can be expressed as

\[
H(z) = \sum_{k=0}^{M-1} h(nM + k)z^{-n} \quad (21)
\]

Where

\[
p_i(n) = h(nM + k), \quad k = 0, 1, 2, \ldots, M-1 \quad (23)
\]

The output of the interpolator can be expressed as

\[
Y(z) = H(z)X(z) \quad (24)
\]

The polyphase structure for FIR interpolator is shown in Fig 6. This structure is the efficient implementation of the interpolation filter since filtering operation in the Polyphase

\[
\begin{align*}
\text{Input:} \quad & x(n) \\
\text{Output:} \quad & y(m) \\
\text{Sampling Rate:} \quad & L \quad \text{upsamplers} \\
\end{align*}
\]

\[
\begin{align*}
p_1(n) & = 1, L-1, L-2, \ldots, 0 \\
p_2(n) & = 0, L-1, L-2, \ldots, 1 \\
\end{align*}
\]

The output $Y(z)$ is

\[
Y(z) = H(z)X(z) \quad (24)
\]

The output can be expressed as

\[
Y(z) = \sum_{k=0}^{M-1} h(nM + k)z^{-n} \quad (25)
\]

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\[
\begin{align*}
\text{Input:} \quad & x(n) \\
\text{Output:} \quad & y(m) \\
\text{Sampling Rate:} \quad & L \quad \text{upsamplers} \\
\end{align*}
\]

\[
\begin{align*}
p_1(n) & = 1, L-1, L-2, \ldots, 0 \\
p_2(n) & = 0, L-1, L-2, \ldots, 1 \\
\end{align*}
\]

Thus for each input sample $x(n)$ each of the $L$ branches of the polyphase network contributes one non-zero output which corresponds to one of the $L$ outputs of the network. The signal representation for the above polyphase interpolator is shown in Fig 7. Since the up-sampled signals in polyphase branches have $L-1$ zero valued samples, the set of upsamplers and delays in the interpolator structure of Fig. 6 can be replaced with the
Fig. 8: Polyphase Structure of an Interpolator with Commutator

The output samples \( y(m) \) are obtained by picking up the filtered samples sequentially at the output sampling rate.

V. MULTISTAGE IMPLEMENTATION OF AN INTERPOLATOR

The interpolator structures discussed above are single stage filters. These structures are efficient for lower order interpolation factors (such as \( L = 2 \) or 3). But for the higher order interpolation factors that are required in the modern digital communication systems like WIMAX (required order of upsampling is 16), WIMAX, etc. single stage implementation is never used. A multistage implementation (Lu and Gupta, 1979; Meyer and Burrus, 1976) of the interpolator is preferred in these applications. The interpolation factor \( L \) is factored into the product of integers i.e. \( L = L_1L_2 \ldots \ldots L_K \) and implemented as a cascade of \( K \) interpolators as shown in Fig. 9. This structure is called the multistage implementation of an interpolator.

\[
\begin{align*}
K_L & \rightarrow H_1(z)H_2(z^L) \cdots \cdots H_K(z^{L_{K-1}}) \\
& \text{(25)}
\end{align*}
\]

The direct form as well as Polyphase structures can be implemented efficiently using multistage implementation which leads to less computational complexity in the implementation of an interpolator.

Multistage implementation of an interpolator results in

- The reduction of computation complexity
- Less storage requirement in the system
- Simplified filter design

Along with these advantages the multistage implementation has some drawbacks:

- The number of stages of the system cannot be increased beyond a certain value. After that the complexity of the system starts increasing.
- Proper control structure is required to handle all stages of the system.

VI. CONCLUSIONS

We have discussed different interpolator structures such as direct form as well as polyphase. The computational efficiency of any interpolator structure is determined by the number of multiplications per output sample as well as by the number of multiplications per second required for the filter. The direct form structure as well as polyphase structure reduces the number of multiplications by a factor of \( L \) as compared to the conventional FIR interpolator structure. The number of multiplications required per output sample for polyphase structure is same as that of direct form but the number of delay elements required in this case is less as compared to single direct structure. The direct form structures have the advantage that they can be easily modified by using symmetry property of FIR filters. The polyphase structures have the advantage that the polyphase filters or polyphase components can be easily realized by using simple and efficient techniques such as convolution method based on FFT.

REFERENCES