Integral Solutions of The Homogeneous Cubic Diophantine Equation with 6 Unknowns $(w^2 + p^2 - z^2)(w - p) = (x + y)R^2$

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Abstract— The homogeneous cubic equation with 6 unknowns represented by $(w^2+p^2-z^2)(w-p)=(x+y)R^2$ is analyzed for its non-zero distinct integer solutions. Employing the transformations and applying the method of factorization, five different patterns of non-zero distinct integer solutions to the above cubic equation are obtained. A few interesting relations between the solutions and special numbers namely Polygonal numbers and Pyramidal numbers are presented.

Key words: cubic equation with 6 unknowns, Homogeneous cubic equation, integral solutions 2010 Mathematics Subject Classification: 11D09

I. INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular cubic Diophantine equations, homogeneous and non homogeneous have aroused the interest of numerous mathematicians since antiquity [1-4]. In this context one may refer [5-9] for various problems on the cubic Diophantine equations come with four variables. However, often we across non-homogeneous cubic equations and as such one may require its integral solution in its most general form. It is towards this end this paper concerns with the problem of determining non-trivial integral solutions of the cubic equation with six unknowns given by

$$(w^2+p^2-z^2)(w-p)=(x+y)R^2$$

II. METHOD OF ANALYSIS

The equation under consideration is

$(w^2+p^2-z^2)(w-p)=(x+y)R^2$ (1)

Introduce the transformations

W=2u + v, p =2u – v, z = v, x = v+1, y = v–1 (2)

Substituting (2) in (1) we get

$v^2 + 8u^2 = R^2$ (3)

A. Pattern: 1

Let $R(a, b) = a^2 + 8b^2$ (4)

Using (4) in (3) we get

$v^2 + 8u^2 = (a^2 + 8b^2)^2$

By method of factorization, we get

$(v+i\sqrt{8}\,u)(v-i\sqrt{8}\,u)=(a+i\sqrt{8}b)^2(a-i\sqrt{8}\,b)^2$ (5)

Equating positive and negative factors, we get

$(v + i\sqrt{8}\,u) = (a + i\sqrt{8}\,b)^2$ (6)

$(v - i\sqrt{8}\,u) = (a - i\sqrt{8}\,b)^2$ (7)

Equating real and imaginary parts, we get

$v = a^2 - 8b^2$ (8)

$u = 2ab$ (9)

Substituting (8) & (9) in (2) the integer solutions of (1) are found to be

$w(a, b) = 4ab + a^2 - 8b^2$

$p(a, b) = 4ab - a^2 + 8b^2$

$z(a, b) = a^2 - 8b^2$

$x(a, b) = a^2 - 8b^2 + 1$

$y(a, b) = a^2 - 8b^2 - 1$

$R(a, b) = a^2 + 8b^2$

1) *Properties:*

(1) 3[w (a,1) p(a,1)+ z(a,1) $-9t_{4,a}$] Nasty number
(2) y(1, b) + p(1,b) $+10gn_b + 11$ Nasty number
(3) w (a, 2) - 2 $t_{3,a}$ ≡ 3 (mod 7)
(4) x(a, 1) - p(a, 1) + $6SO_a - 4t_{3,a}$ ≡ -3 (mod 12)
(5) w(n(n + 1),n + 2)+ p(n(n + 1),(n + 2)) ≡ $48P_n^3$
(6) z(a,1) - Pr_a -gn_a ≡ -1 (mod 3)
(7) [w (a, b) - p(a, b) + R(a, b)-19 $t_{4,b}$] = Difference between two perfect squares

B. Pattern: 2:

Rewrite (3) as $v^2 + 8u^2 = R^2 * 1$ (10)

Write 1 as $1 = \frac{(1+i\sqrt{8})(1-i\sqrt{8})}{9}$ (11)

Substituting (4) & (11) in (10) and by method of factorization, we get

$(v+i\sqrt{8}\,u) = (a+i\sqrt{8}\,b)^2\ \frac{(1+i\sqrt{8})}{3}$ (12)

Equating real and imaginary parts, we get

$v = \frac{1}{3}(a^2 - 16ab - 8b^2)$ (13)

$u = \frac{1}{3}(a^2 + 2ab - 8b^2)$ (14)

As our interest is on finding integer solutions, it is seen that the values of u and v are integers for suitable choice of the parameters a and b.

Putting a = 3A and b = 3B in (13) & (14) we get

$v = (3A^2 - 48AB - 24B^2)$ (15)

$u = (3A^2 + 6AB - 24B^2)$ (16)

Substituting (15) & (16) in (2) the integer solutions of (1) are found to be

$w(A, B) = 9A^2 - 36AB - 72B^2$

$p(A, B) = 3A^2 + 60AB - 24B^2$

$z(A, B) = 3A^2 - 48AB - 24B^2$

$x(A, B) = 3A^2 - 48AB - 24B^2 + 1$

$y(A, B) = 3A^2 - 48AB - 24B^2 - 1$

$R(A, B) = 9A^2 + 72B^2$

1) *Properties:*

(1) w (A, 1) – $t_{20,A}$ ≡ $12 (mod\ 28)$
(2) w (2,B) + 144$t_{3,B}$ =36
(3) z (A, 1) – 6$t_{3,A}$ ≡ $27 (mod\ 51)$
(4) 3z [n (n+1), n + 2] - w [n(n+1),n + 2] = $648P_n^3$
(5) 5. z(A, 1) + x (A, 1) + 47 Nasty number.
(6) p (A,1) - y(A,1) - 6gn_A -1 Nasty number.
(7) R (A, 1) + 3p (A, 1) + 6$t_{4,A}$ Nasty number.

C. Pattern: 3:

In (10) write 1 as

$1 = \frac{(1+i6\sqrt{8})(1-i6\sqrt{8})}{289}$ (17)

Substituting (4) &(17) in (10) by employing the method of factorization and equating the positive and negative factors we get

$$(v + i\sqrt{8}\,u) = (a + i\sqrt{8}\,b)^2 \frac{(1+i6\sqrt{8})}{17} \quad (18)$$

Equating real and imaginary parts in (18), we get

$$v = \frac{1}{17}(a^2 - 96ab - 8b^2) \quad (19)$$

$$u = \frac{1}{17}(6a^2 + 2ab - 48b^2) \quad (20)$$

As our interest is on finding integer solutions, it is seen that the values of u and v are integers for suitable choice of the parameters a and b.

Putting a = 17A and b = 17B in (19) & (20), we get the corresponding integer solutions of (1) are found to be

w (A, B) = $221A^2 - 1564AB - 1768B^2$
p (A, B) = $187A^2 + 1700AB - 1496B^2$
z (A, B) = $17A^2 - 1632AB - 136B^2$
x (A, B) = $17A^2-1632AB-136B^2+1$
y (A, B) = $17A^2-1632AB-136B^2-1$
R (A, B) = $289A^2 + 2312B^2$

1) Properties:

(1) 3[x(A, B) – y(A,B)] Nasty Number
(2) w(A, 1) - p(A,1) - 34Pr_A
(3) z(A,1) -34 $t_{3,A} \equiv -136 (mod\ 1649)$
(4) w(A, 1) + R(A,1) – 510 $\equiv 153 (mod\ 391)$

D. Pattern: 4

Rewrite (10) as $R^2 - 8u^2 = v^2 * 1$ (21)
Take $v = a^2 - 8b^2$ (22)
Write 1 as $1 = (3 + \sqrt{8})(3 - \sqrt{8})$ (23)
Substituting (22) & (23) in (21), we get
$R^2 - 8u^2 = (a^2 - 8b^2)^2 (3 + \sqrt{8})(3 - \sqrt{8})$

By method of factorization and by equating positive and negative factors, we get

$R + \sqrt{8}u = (a + \sqrt{8}\ b)^2 (3 + \sqrt{8})$ (24)

Equating rational and irrational parts, we get

$R = 3a^2 + 16ab + 24b^2$ (25)
$u = a^2 + 6ab + 8b^2$ (26)

Substituting (22) & (26) in (2) the integer solutions of (1) are found to be

w (a, b) = $3a^2 + 12ab + 8b^2$
p (a, b) = $a^2 + 12ab + 24b^2$
z (a, b) = $a^2 - 8b^2$
x (a, b) = $a^2 - 8b^2 + 1$
y (a, b) = $a^2 - 8b^2 - 1$
R (a, b) = $3a^2 + 16ab + 24b^2$

1) Properties:

(1) 12w (a, 1) -12 p (a, 1) - 4 Nasty number
(2) w (a, a)+p (a, a)+z(a, a)+7$t_{4,a}$Nasty number
(3) 3[x (a, b) – y (a, b)] Nasty number
(4) 5[R (a, a) + z (a, a)] Nasty number
(5) 4Pr_a – x (a,1) – w (a,1) ≡ 1(mod 8)

E. Pattern: 5:

$17 + 6\sqrt{8})(17 - 6\sqrt{8})$ (27)
Substituting (27) in (21) we get
$R^2 - 8u^2 = (a^2 - 8b^2)^2 (17 + 6\sqrt{8})(17 - 6\sqrt{8})$
By method of factorization and equating rational and irrational parts, we get
$R = 17a^2 + 96ab + 136b^2$
$u = 6a^2 + 34ab + 48b^2$

Therefore the integer solutions of (1) becomes
w (a, b) = $13a^2 + 68ab + 88b^2$
p (a, b) = $11a^2 + 68ab + 104b^2$
z (a, b) = $a^2 - 8b^2$
x (a, b) = $a^2 - 8b^2 + 1$
y (a, b) = $a^2 - 8b^2 - 1$
R (a, b) = $17a^2 + 96ab + 136b^2$

1) Properties:

(1) w (a, 1) - p(a, 1) + 4$T_{4,a}$ + 16 Nasty number
(2) 2[w (a, 1) - p(a, 1)+ z(a, 1) - 6] Nasty number
(3) R $(a, 1) - 34T_{3,a} \equiv 915$ (mod 1247)
(4) 3[x (a, 1) – y(a, 1)] Nasty number

III. CONCLUSION

To conclude, one may search for other patterns of solutions and their corresponding properties.

REFERENCES

[1] Carmicheal R. D., The Theory of Numbers and Diophantine Analysis, Dover Publication, New York, (1959).
[2] Dickson, L.E.,History of Theory of numbers,Vol.2, Chelsea publishing company, New York, 1952.
[3] Mordell, L.J., Diophantine equations, Academic press, London (1969).
[4] Telang S.G.,Number Theory,Tata Mc Graw Hill Publishing Co., New Delhi,(1996).
[5] Gopalan M.A. and Premalatha.S, Integral Solutions of $(x + y)(xy + w^2) = 2(k^2 + 1) z^3$, Bulletin of Pure and Applied Sciences, Vol.29E,No:2, 197-202,2009.
[6] Gopalan M.A. and Pandichelvi.V, Remarkable Solutions of cubic equation with four unknowns $x^3 + y^3 + z^3 = 28(x + y + z) w^2$, Antarctica. J. Math, Vol.7, No.4, 393 – 401,2010.
[7] Gopalan M.A., sivagami.B., Integral Solutions of homogeneous cubic Equation with four Unknowns $x^3 + y^3 + z^3 = 3xyz+2(x+y) w^2$, Impact J.Sci.Tech., Vol.4, No.3,53–60,2010.
[8] Gopalan M.A. and Geetha.K, Observation on cubic Equation with four Unknowns $x^3 + y^3 + xy(x+y) = z^3 + 2(x+y) w^2$, International Journal of Pure And Applied Mathematical Sciences,Vol.6, No.1, 25 – 30,2013.
[9] Gopalan M.A. Vijayalakshmi.S and Sumathi.G, On the homogeneous cubic Equation with four Unknowns $x^3+y^3=14z^3-3w^2(x+y)$, Discovery,Vol.2, No.4, October 17–19,2012.