Integral Points on the Homogenous Cone $x^2 + y^2 = 10z^2$

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Abstract— The ternary homogenous quadratic Diophantine equation given by $x^2+y^2 = 10z^2$ is analyzed for finding its non-zero distinct integral solutions. Five different patterns of integer solutions are presented. A few interesting relations between the solutions and special number patterns are given.

Key words: homogenous quadratic, integer solutions, special number 2010 Mathematics Subject Classification: 11D09

NOTATIONS USED:

P_n^m -Pyramid number of rank n with size m.

$T_{m,}$ - Polygonal number of rank n with size m.

I. Introduction

The Ternary Quadratic Diophantine Equation offers an unlimited field for research because of their variety (1-2) For an extensive review of various problems, one may refer (3-6). This communication concerns with yet another interesting Ternary Quadratic equation $x^2 + y^2 = 10z^2$ representing a homogenous cone for determining it's infinitely may non-zero integral solutions. Also a few interesting relations among the solutions have been presented.

II. Method of Analysis

The ternary quadratic equation to be solved for its non-zero integer solutions is

$x^2 + y^2 = 10z^2$ (1)

Assume $z(A, B) = A^2+B^2$, where $A, B > 0$ (2)

We illustrate below five different patterns of non-zero distinct integer solutions to (1)

A. Pattern-I:

Write 10 as

$10 = (3+i)(3-i)$ (3)

Substituting (2) and (3) in (1), employing the method of factorization, define

$(x + iy)(x - iy) = (3+i)(3-i)(A + iB)^2 (A - iB)^2$

Equating real and imaginary parts, we get

$x = x(A, B) = 3A^2 - 3B^2 - 2AB$ (4)

$y = y(A,B) = A^2 - B^2 + 6AB$ (5)

Thus (2), (4) and (5) represents non-zero distinct integral solutions of (1) in two parameters.

As our interest is on finding integer solutions, we choose A and B suitably so that the values of x, y and z are in integers. In what follows the values of A, B and the corresponding integer solutions are exhibited.

1) Case 1:

Let A = 2A, B = 2B

The corresponding solutions of (1) are

$x = x(A, B) = 12A^2 - 12B^2 - 8AB$

$y = y(A, B) = 4A^2 - 4B^2 + 24AB$

$z = z(A, B) = 4A^2 + 4B^2$

2) Properties:

(1) $x(A, 1) - t_{26,A} \equiv 0 \pmod 3$
(2) $3y(A, A+1) - x(A, A+1) = 160\, t_{3,A}$
(3) $y(A, 1) - t_{10,A} \equiv -1 \pmod 3$
(4) $6\{z(A, A(A+1)) - 16\, t_{3,A}^2\}$ a Nasty number

3) Case 2:

Let A = 2A, B = B

The corresponding solutions of (1) are

$x = x(A, B) = 12A^2 - 3B^2 - 4AB$

$y = y(A, B) = 4A^2 - B^2 + 12AB$

$z = z(A, B) = 4A^2 + B^2$

4) Properties:

(1) $x(A(A+1),(A+2)) - 3y(A(A+1),(A+2)) + 240\, P_A^3 = 0$
(2) $x(A, A+1) - 3y(A, A+1) - 80\, t_{3,A} = 0$
(3) $x(A, 1) - t_{26,A} \equiv -3 \pmod 7$
(4) $2\{x(A, A(A+1)) + 12\, t_{3,A}^2 + 8\, P_A^5\}$ a Nasty number

5) Case 3:

Let A = 2A, B = 2A+1

The corresponding solutions of (1) are

$x = x(A, B) = -8A^2 - 16A - 3$

$y = y(A, B) = 24A^2 + 8A - 1$

$z = z(A, B) = 8A^2 + 4A + 1$

6) Properties:

(1) $z(A(A+1), 1) - 32\, t_{3,A}^2 - 8t_{3,A} = 1$
(2) $y(A, A) - z(A, A) - 8t_{4,A} \equiv 0 \pmod 3$
(3) $y(A, 1) - t_{50,A} \equiv -1 \pmod{31}$

B. Pattern – II:

Instead of (3), write 10 as

$10 = (1+3i)(1-3i)$ (6)

Following the procedure as presented in pattern 1, the corresponding values of x and y are

$x = x(A, B) = A^2 - B^2 - 6AB$ (7)

$y = y(A, B) = 3A^2 - 3B^2 + 2AB$ (8)

Thus (2), (7) and (8) represents non-zero distinct integral solutions of (1) in two parameters.

1) Properties:

(1) $3x(A, A+1) - y(A, A+1) + t_{32,A} = 0$
(2) $x(A, A)$ a Nasty number
(3) $y(A, A) - 2\, t_{4,A} = 0$
(4) $y(2A, A) - t_{28,A} \equiv 0 \pmod 2$
(5) $z(A, A) - 2\, t_{4,A} = 0$
(6) $y(A,1) - t_{8,A} \equiv -3 \pmod 4$
(7) $2\{y(A, A(A+1)) + 12\, T_{3,A}^2 - 4\, P_A^5\}$
(8) $6\{x(A, A(A+1)) + 4\, T_{3,A}^2 + 12\, P_A^5\}$

C. Pattern – III:

(1) is written in the form of ratio as

$\frac{x+y}{3z+y} = \frac{3z-y}{x-z} = \frac{A}{B}$, B ≠ 0,

which is equivalent to the system of equations

Bx – Ay + (B – 3A) z = 0 (9)

Ax + By - (A + 3B) z = 0 (10)

Applying the method of cross multiplication the integer solutions of (1) are given by

x = x (A, B) = A^2 - B^2 + 6AB (11)

y = y (A, B) = $3A^2$ - $3B^2$ + 2AB (12)

z = z (A, B) = A^2 + B^2 (13)

1) *Properties:*

(1) x (A, A+1) – z (A, A+1) + $t_{6,A}$ - 12 $t_{3,A}$ ≡ 2 (mod 5)

(2) x (A(A+1), A+2) – z (A(A+1), A+2) + $t_{6,A}$ -36 P_A^3 ≡ 8 (mod 9)

(3) 3x (A^2, A+1) + y (A^2, A+1) - 40 P_A^5 =0

(4) y (A, 1) + $t_{8,A}$ ≡ 0 (mod 3)

(5) x (A, 1) - $t_{4,A}$ ≡ -1 (mod 6)

(6) z (A,A) – $t_{4,A}$ = 0

(7) y (2A,A) + $t_{12,A}$ ≡ 0 (mod 2)

(8) y(A,1) + $t_{8,A}$ ≡ 0 (mod 3)

(9) 6{x(A,A(A+1) + 4 $T_{3,A}^2$ - P_A^5 } a Nasty number

(10) 6{z(A,A(A+1) - 4 $T_{3,A}^2$ } a Nasty number

D. *Pattern – IV:*

(1) is written as

($10z^2$ –y^2) = x^2 = x^2 * 1 (14)

Assume x (A, B) = 10 A² - B² (15)

Write 1 as $1 = (\sqrt{10}+3)(\sqrt{10}-3)$ (16)

Substituting (15) and (16) in (14) and applying the method of factorization, define

(√10 z + y) = $(\sqrt{10}\,A + B)^2$ (√10 + 3)

Equating rational and irrationals parts, we have

y = y (A, B) = $30A^2$ + $3B^2$ + 20AB (17)

z = z (A, B) = $10A^2$ + B^2 + 6AB (18)

1) *Properties:*

(1) y (A,A+1) – z (A,A+1) - 4$t_{3,A}$ = 0

(2) y (A, A(A+1)) 80 $t_{4,A}$ - 12 $T_{3,A}^2$ - 40 P_A^5 =0

(3) x (A, A) -9$T_{4,A}$ = 0

(4) y (A, 1) – $t_{62,A}$ ≡ 3(mod 49)

(5) z (A, 1) – $t_{22,A}$ ≡ 1(mod 3)

(6) x (A, 2A) – $t_{14,A}$ ≡ 0(mod 5)

(7) z (A,A) – $t_{36,A}$ ≡ 0(mod 4)

(8) x (A, 1) + z(A, 1) - $t_{42,A}$ ≡ 0 (mod 5)

(9) x(1, B) –z (1, B) + $t_{6,B}$ ≡ 0 (mod 7)

(10) x(A,A) – $t_{20,B}$ ≡ -1 (mod 2)

E. *Pattern – V:*

Assume x (A, B) = 10 A^2 –B^2 (19)

Substituting (19) in (1) and applying the method of factorization, define

$(\sqrt{10}\,A + B)^2$ = (√10 z + y)

Equating rational and irrational we get,

y = y (A, B) = $10A^2$ + B^2 (20)

z = z (A, B) = 2AB (21)

As our interest is on finding integer solutions, We choose A and B suitably so that the values of x, y and z are in integers. In what follows the values of A, B and the corresponding integer solutions are exhibited.

1) *Case 1:*

Let A = 2A, B = 2B

The corresponding solutions of (1) are

x = x (A, B) = $40A^2$ - $4B^2$

y = y (A, B) = $40A^2$ + $4B^2$

z = z (A, B) = 8AB

2) *Properties:*

(1) y (1, B) – x (1, B) - z (1, B)-16$t_{3,A}$ ≡ 0(mod 4)

(2) x (B, 1) + y (B, 1) - z (B, 1)- $t_{162,A}$ ≡ 0(mod 87)

(3) x (A, 1) + z (B, 1) – $t_{82,A}$ ≡ -4(mod 47)

(4) 6{y (A(A+1), A) - 160 $T_{3,A}^2$ } a Nasty number.

3) *Case 2:*

Let A = 2A, B = 2B+1

The corresponding solutions of (1) are

x = x (A, B) = $40A^2$ - $4B^2$- 4B - 1

y = y (A, B) = $40A^2$ + $4B^2$ + 4B + 1

z = z (A, B) = 2A + 4AB

4) *Properties:*

(1) y (A, B) – x (A, B) – $t_{18,B}$ ≡ 2(mod 3)

(2) y (1, B) – $t_{10,B}$ ≡ -1(mod 7)

5) *Case 3:*

Let A = 2A, B = B

The corresponding solutions of (1) are

x = x (A, B) = $40A^2$ - $4B^2$

y = y (A, B) = $40A^2$ + $4B^2$

z = z (A, B) = 4AB

6) *Properties:*

(1) x (A, 1) – $t_{82,A}$ ≡ -1(mod 3)

(2) z (A, A)- $t_{10,A}$ ≡ 0(mod 8)

(3) .x (A, A(A+1)) + y (A, A(A+1)) - 40 $t_{4,A}$ = 0

(4) z(A, A(A+1)) - 8 P_A^5 = 0

III. CONCLUSION

In this paper we have presented six different patterns of non-zero distinct integer solutions of the homogeneous equations given by $X^2 + Y^2 = 10Z^2$. To conclude one may search for other patterns of solution and their corresponding properties.

REFERENCES

[1] Dickson,L.E., History of Theory of Numbers,Vol.2, Chelsea Publishing company, New York, 1952

[2] Mordell,L.J., Diophantine equations, Academic press, NewYork,1969

[3] S.Divya,M.A.GopalanandS.Vidhyalakshmi.,”Lattie points on the homogeneous cone3(X^2 +Y^2)-2XY= $44Z^2$”Jamal Academic Research Journal (190-195),2013

[4] Gopalan, M.A., Manju somanath, Integral solution of ternary quadratic Diophantine equation xy + yz = zx Antarticaj,math,1-5,5(1),2008

[5] Gopalan,M.A.,Kalinga Rani,J.,”Observation on the Diophantine equa equation,y^2=Dx^2+z,Impact J.sci tech;Vol(2),91-95,2008

[6] Gopalan,M.A.,Sangeetha, G.,Observations on y^2=$3x^2$ -$2z^2$ Antarctica J.Math,9(4),359-362,2012

