Integral Points on the Homogenous Cone $x^2 + y^2 = 10z^2$

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Abstract— The ternary homogenous quadratic Diophantine equation given by $x^2+y^2 = 10z^2$ is analyzed for finding its non-zero distinct integral solutions. Five different patterns of integer solutions are presented. A few interesting relations between the solutions and special number patterns are given.

Key words: homogenous quadratic, integer solutions, special number patterns

NOTATIONS USED:

- $P_n^m$: Pyramid number of rank $n$ with size $m$.
- $T_m$: Polygonal number of rank $n$ with size $m$.

I. INTRODUCTION

The Ternary Quadratic Diophantine Equation offers an unlimited field for research because of their variety and many interesting relations among the solutions have been presented.

II. METHOD OF ANALYSIS

The ternary quadratic equation to be solved for its non-zero integer solutions is

$$x^2 + y^2 = 10z^2$$

(1)

Assume $z (A, B) = A^2+B^2$, where $A, B > 0$

(2)

We illustrate below five different patterns of non-zero distinct integer solutions to (1).

A. Pattern-I:

Write 10 as

$$10 = (3+i) (3-i)$$

(3)

Substituting (2) and (3) in (1), employing the method of factorization, define

$$x + iy) (x - iy) = (3+i) (3-i) (A + iB)^2 (A - iB)^2$$

Equating real and imaginary parts, we get

$$x = x (A, B) = 3A^2 - 3B^2 - 2AB$$

(4)

$$y = y (A, B) = A^2 - B^2 + 6AB$$

(5)

Thus (2), (4) and (5) represents non-zero distinct integral solutions of (1) in two parameters.

As our interest is on finding integral solutions, we choose $A$ and $B$ suitably so that the values of $x$, $y$ and $z$ are in integers. In what follows the values of $A$, $B$, and the corresponding integer solutions are exhibited.

I) Case I:

Let $A = 2A$, $B = 2B$

The corresponding solutions of (1) are

$$x = x (A, B) = 12A^2 - 12B^2 - 8AB$$

(6)

$$y = y (A, B) = 4A^2 - 4B^2 + 24AB$$

(7)

$$z = z (A, B) = 4A^2 + 4B^2$$

(8)

II) Properties:

1) $x (A, 1) - t_{6, A} \equiv 0 \pmod{3}$

2) $3y (A, A+1) - x (A, A+1) = 160 t_{3, A}$

3) $y (A, 1) - t_{1, A} \equiv -1 \pmod{3}$

4) $6z (A, A(A+1)) - 16 t_{3, A}^2 \}$ a Nasty number

III) Case II:

Let $A = 2A$, $B = B$

The corresponding solutions of (1) are

$$x = x (A, B) = 2A^2 - 3B^2 - 4AB$$

(9)

$$y = y (A, B) = 4A^2 - B^2 + 12AB$$

(10)

$$z = z (A, B) = 4A^2 + B^2$$

(11)

IV) Properties:

1) $x (A(A+1), (A+2)) - 3y (A(A+1), (A+2)) + 240 P_A^3 = 0$

2) $x (A, A+1) - 3y (A, A+1) - 80 t_{4, A} = 0$

3) $x (A, 1) - t_{8, A} \equiv -3 \pmod{7}$

4) $2[x (A, A(A+1)) + 12 t_{3, A}^2 + 8 P_A^5 \} \}$ a Nasty number

V) Case III:

Let $A = 2A$, $B = 2A+1$

The corresponding solutions of (1) are

$$x = x (A, B) = -8A^2 - 16A - 3$$

(12)

$$y = y (A, B) = 24A^2 + 6A - 1$$

(13)

$$z = z (A, B) = 8A^2 + 4A + 1$$

(14)

VI) Properties:

1) $z (A(A+1), 1) - 32 t_{3, A}^2 - 8 t_{3, A} = 1$

2) $y (A, A) - 3z (A, A) - 8t_{3, A} = 0 \pmod{3}$

3) $y (A, 1) - t_{8, A} \equiv -1 \pmod{31}$

B. Pattern – II:

Instead of (3), write 10 as

$$10 = (1+3i)(1-3i)$$

(6)

Following the procedure as presented in pattern 1, the corresponding values of $x$ and $y$ are

$$x = x (A, B) = A^2 - B^2 - 6AB$$

(7)

$$y = y (A, B) = 3A^2 - 3B^2 + 2AB$$

(8)

Thus (2), (7) and (8) represents non-zero distinct integral solutions of (1) in two parameters.

I) Properties:

1) $3x (A, A+1) - y (A, A+1) + t_{32, A} = 0$

2) $x (A, A)$ a Nasty number

3) $y (A, A) - 2 t_{4, A} = 0$

4) $y (2A, A) - 32 t_{3, A} = 0 \pmod{2}$

5) $x (A, A) - 2 t_{4, A} = 0$

6) $y (A, 1) - t_{6, A} \equiv -3 \pmod{4}$

7) $2[y (A, A(A+1)) + 12 T_{3, A}^2 - 4 P_A^5 \} \}$

8) $6[x (A, A(A+1)) + 4 T_{3, A}^2 + 12 P_A^5 \} \}$

C. Pattern – III:

(1) is written in the form of ratio as

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x + y = 3z - y = \frac{A}{B}, \quad B \neq 0,
\frac{3z + y}{x - z} = \frac{A}{B}

which is equivalent to the system of equations

Bx - Ay + (B - 3A)z = 0 \quad \text{(9)}
Ax + By - (A + 3B)z = 0 \quad \text{(10)}

Applying the method of cross multiplication the integer solutions of (1) are given by

x = x(A, B) = A^2 - B^2 + 6AB \quad \text{(11)}
y = y(A, B) = 3A^2 - 3B^2 + 2AB \quad \text{(12)}
z = z(A, B) = A^2 + B^2 \quad \text{(13)}

1) Properties:

1) Case 1:
Let A = 2A, B = 2B
The corresponding solutions of (1) are

x = x(A, B) = 40A^2 - 4B^2 \quad \text{(14)}
y = y(A, B) = 40A^2 + 4B^2 \quad \text{(15)}
z = z(A, B) = 8AB \quad \text{(16)}

2) Properties:

(1) y(1,B) - x(1,B) - z(1,B) - 16t_{1,3A} \equiv 0 \mod 4
(2) y(1,B) + y(1,B) - z(1,B) - t_{621,A} \equiv 0 \mod 87
(3) x(1,1) + z(1,1) - t_{621,A} \equiv -4 \mod 47
(4) 6y(A(A+1), A) - 160T_{3,A}^2 \equiv 0 \mod 5

3) Case 2:
Let A = 2A, B = 2B + 1
The corresponding solutions of (1) are

x = x(A, B) = 40A^2 - 4B^2 - 4B - 1 \quad \text{(17)}
y = y(A, B) = 40A^2 + 4B^2 + 4B + 1 \quad \text{(18)}
z = z(A, B) = 2A + 4AB \quad \text{(19)}

4) Properties:

(1) y(A,B) - x(A,B) - t_{10,B} \equiv 2 \mod 3
(2) y(1,B) - t_{10,B} \equiv -1 \mod 7

5) Case 3:
Let A = 2A, B = B
The corresponding solutions of (1) are

x = x(A, B) = 40A^2 - 4B^2 \quad \text{(20)}
y = y(A, B) = 40A^2 + 4B^2 \quad \text{(21)}
z = z(A, B) = 4AB \quad \text{(22)}

6) Properties:

(1) x(1) \equiv t_{621,A} \equiv -1 \mod 3
(2) z(A,A) - t_{10,A} \equiv 0 \mod 8
(3) x(A, A(A+1)) + y(A, A(A+1)) - 40t_{621,A} \equiv 0 \mod 87
(4) z(A, A(A+1)) - 8P_A^5 \equiv 0 \mod 47

III. CONCLUSION

In this paper we have presented six different patterns of non-zero distinct integer solutions of the homogeneous equations given by X^2 + Y^2 = 10Z^2. To conclude one may search for other patterns of solution and their corresponding properties.

REFERENCES