Propagation Behaviour of Solid Dielectric Rectangular Waveguide

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Abstract— For frequencies above 30 ghz, increasing skin depth losses in metal requires that low loss structures be made without the use of metallic materials. Hence, the importance of pure dielectrics waveguides for carrying large bandwidth signals is established. The only unexploited spectral region, Terahertz band, is now being actively explored. Moreover, metallic waveguides or antennas are dangerous when the application involves ionized gas i.e. Plasma or when there is a risk that the antenna or waveguide can be exposed to plasma. Dielectric waveguides might be the only viable solution. Here, an analytical theory has been developed for finding out the modal characteristics of a solid dielectric waveguide in guided and leaky modes.

Key words: Dispersion characteristics; leaky mode; guides modes; dielectric; TE&TM modes.

I. INTRODUCTION

When a waveguide is excited, various higher order modes can be generated along with the fundamental mode. Up till now, the propagation characteristics of these higher order modes have yet not been formulated and have been out of the spotlight. Here, an expression has been developed for finding out the modal characteristics of a solid dielectric waveguide. The variables of field equations are separated into three space coordinates and substituted in the Maxwell’s equation in potential form. Boundary conditions are applied to these wherein the fields at the boundaries of the dielectric waveguide are matched. This gives us a set of transcendental equation for transverse propagation constant which is iteratively solved using MATLAB to find a solution.

II. METHODS USED TO EVALUATE THE PROPAGATION CONSTANT

There are many methods to evaluate the propagation constant of a rectangular waveguide, but we would concentrate on the approximate methods, as they are less complex and hence time and resource saving. The two different methods that have been taken into consideration here are

- Marcatili’s approach
- Circular harmonics approach

Out of the two methods, Marcatili’s approach is relatively simple and easy to put into operation. Circular harmonics is a relatively complex but converges faster than Marcatili’s method.[1]

III. PROBLEM FORMULATION

The problem was formulated using the method below:

- Separation of variables is used
- Substituted in the Maxwell’s equation in potential form
- Boundary conditions are applied for rectangular waveguide
- Gives us set of transcendental equation
- Substituted in the characteristic equation and the propagation constant in z-direction calculated
- Dispersion relationship obtained
- Same methodology would be extended for the leaky modes.

Fig. 1 Cross section of a rectangular waveguide

Fig. 2: Regions considered in the Marcatili’s approach

The priori assumption, for the analysis of the waveguide using Maractili’s approach, is that, the propagation is in guided mode, means that almost all the power is contained within the waveguide core. Very less power is in the cladding region. Hence, the boundary conditions are applied keeping in mind the fields that would carry the guided power. Very less power is in the corner regions of the guide. Marcatili formulated the solution to the problem by matching the fields at the edges of the guide and ignoring the power at the corners.[2]

While considering the circular harmonics approach, we think on the cylindrical coordinates, as shown in Fig 3. In both the cases, it is considered that the rectangular core with permittivity $\varepsilon_1$ is surrounded by the infinite medium.
with permittivity $\varepsilon_2$. Both the mediums are isotropic with permeability $\mu_0$. The propagation is in z+ direction.\[3\]

![Image](image_url)

**Fig. 3: Dimensions and coordinates for circular harmonics approach \[3\]**

**IV. FIELD EQUATIONS AND SOLUTIONS FOR GUIDED MODES**

While considering the Marcatilli’s method, the rectangular coordinate system is considered. Maxwell’s equations are:

\[\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}\]
\[\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\]
\[\nabla \cdot \mathbf{B} = 0\]
\[\nabla \cdot \mathbf{D} = \rho_v\]

On solving these equations, we get Helmholtz equations,

\[\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0\]
\[\nabla^2 \mathbf{H} + k^2 \mathbf{H} = 0\] \[4.5\]
\[4.6\]

Now, as we are dealing with dielectric materials, we do not have pure TE and TM modes. So when the limit electric field is parallel to x axis, we call it $E_x$ mode and when the limit electric field is parallel to y axis, we call it $E_y$ modes.

Now moving further, applying the variable separation method, suppose we have $E_y$ mode, we get the field equations as,

\[E_x = \frac{1}{j\omega \mu} \frac{\partial H_z}{\partial y}\]
\[E_y = \frac{1}{j\omega \varepsilon} \frac{\partial H_x}{\partial x} - \frac{\beta}{\omega} H_x\]
\[E_z = \frac{1}{j\omega \varepsilon} \frac{\partial H_x}{\partial y}\]
\[H_x = \frac{1}{j\omega \mu} \left( \frac{\partial E_y}{\partial y} + j\beta E_y \right)\]

\[4.7\]
\[4.8\]
\[4.9\]
\[4.10\]

And now we do suppose the field equations of waveguide regions as,

\[E_y^{(1)} = A_x \cos(k_{x1} x + \alpha) \cos(k_{y1} y + \gamma)\]
\[E_y^{(2)} = A_x \cos(k_{x2} x + \alpha) e^{-jk_{y2} y}\]
\[E_y^{(3)} = A_x \cos(k_{y3} y + \gamma) e^{-jk_{x3} x}\]
\[E_y^{(4)} = A_x \cos(k_{x4} x + \alpha) e^{-jk_{y4} y}\]
\[E_y^{(5)} = A_x \cos(k_{y5} y + \gamma) e^{-jk_{x5} x}\]

\[k_x^2 + k_y^2 + \beta^2 = k_v^2 = \omega^2 \mu \varepsilon_v\]

Where, $v = 1, 2, 3, 4, 5$ \[2\]

Applying boundary conditions,

\[H_x^{(1)} = H_x^{(2,4)}\]
\[E_z^{(1)} = E_z^{(1,2,3,4)}\]
\[H_z^{(1)} = H_z^{(1,2,3,4)}\]
\[E_y^{(1)} = E_y^{(1,3)}\]

And assuming that,

For matching boundaries between regions ‘1’ & ‘2’, ‘4’

\[k_{x1} = k_{x2}, 4 = k_x\]
\[k_{y2} = k_{y4}\]

And for matching boundaries between regions ‘1’ & ‘3’, ‘5’

\[k_{y1} = k_{y3,5} = k_y\]
\[k_{x2} = k_{x4}\]

We get the dispersion relation as

\[\tan\left( k_{y1} \frac{b}{2} \right) = j \frac{\varepsilon_1 k_{y2,4}}{\varepsilon_{2,4} k_y}\]
\[\tan\left( k_{x1} \frac{a}{2} \right) = j \frac{\varepsilon_1 k_{x3,5}}{\varepsilon_{3,5} k_x}\]

\[4.23\]
\[4.24\]

This gives the modal solution as,

\[k_y b = m - \tan^{-1}\left( \frac{\varepsilon_2 k_y}{\varepsilon_1 \sqrt{k_x^2 - k_y^2 - k_v^2}} \right)\]
\[4.25\]
where \( m \) and \( n \) are arbitrary integers indicating the order of propagating modes.

On considering the circular harmonics approach, we do consider the cylindrical coordinate system, as shown in figure below.

Considering the Helmholtz equation for cylindrical coordinates

\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial E}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 E}{\partial \varphi^2} + \frac{\partial^2 E}{\partial z^2} + k^2 E = 0
\]  

\[(4.26)\]

Fig, 4 : Cylindrical coordinates [5]

And similar for nanetic field also, And applying the variable separation, we land up with

\[
E = \sum_n \int f_n(k_n) B_n(k_n, \varphi) h(n\varphi) h(k_n, z) dk_n
\]

\[
E = \sum_n g_n(k_n) B_n(k_n, \varphi) h(n\varphi) h(k_n, z) dk_n
\]

\[(4.27)\]

Where, \( fn \) and \( gn \) are determined from boundary conditions.

\( B_n(k_n, \varphi) \) is the bessel function. We consider the type of bessel function in accordance to the field behavior.

Supposing the field equations as,

For core region,

\[ E_{z1} = \sum_{n=0}^{\infty} a_n j_n(hr) \sin(n\theta + \varphi_n) \exp[i(k_nz - \omega t)] \]

\[ H_{z1} = \sum_{n=0}^{\infty} b_n j_n(hr) \sin(n\theta + \psi_n) \exp[i(k_nz - \omega t)] \]

\[(4.28)\]

For outside core,

\[ E_{z0} = \sum_{n=0}^{\infty} c_n K_n(hr) \sin(n\theta + \varphi_n) \exp[i(k_nz - \omega t)] \]

\[ H_{z0} = \sum_{n=0}^{\infty} d_n K_n(hr) \sin(n\theta + \psi_n) \exp[i(k_nz - \omega t)] \]

\[(4.29)\]

\( h = (k_1^2 - k_z^2)^{\frac{1}{2}} \) and \( p = (k_x^2 - k_0^2)^{\frac{1}{2}} \)

\( k_1 = \omega(\varepsilon_0) \)

\( k_0 = \omega(\varepsilon_0) \)

Next is to find the equations for fields \( E_0, H_\theta, E_r, H_r \)

For matching the boundaries, we need the tangential fields. So,

\[ E_t = \pm (E_r \sin \theta + E_\theta \cos \theta) \]

\[ E_t = \pm (-E_r \cos \theta + E_\theta \sin \theta) \]

\[ \theta - \theta_c < \pi - \theta < + \theta \]

\[ \Pi + \theta_c < \theta < - \theta_c \]  

\[(4.30)\]

Apply the similar to the magnetic field equations and then consider the boundary conditions,

\[ E_{z1} = E_{z0} \]

\[ \mu_1 H_{z1} = \mu_0 H_{z0} \]

\[ E_{t1} = E_{t0} \]

\[ \mu_1 H_{t1} = \mu_0 H_{t0} \]

\[(4.31)\]

We come up with the solution as,

Longitudinal field components:

\( E_L = E_C \)

\( H_L = H_D \)

Transverse field components:

\( E_T + E_B = E_{TC} + E_{TD} \)

\( H_T + H_B = H_{TC} + H_{TD} \)

\[(4.32)\]

\( D = \begin{bmatrix} E_L & 0 & -E_C & 0 \\ 0 & H_T & 0 & -H_D \\ E_B & E_T & -E_{TC} & -E_{TD} \\ H_B & H_T & -H_{TC} & -H_{TD} \end{bmatrix} \)

\[(4.33)\]

Solving \( \text{Det}(D)=0 \), we get the value for \( k_z \)

V. SOLUTION FOR LEAKY MODES

Now, for leaky modes, just replacing the Bessel function , by Hankel function of the second type, and applying the boundary conditions to the field equations gives the \( k_z \) which is less than \( k \), hence a fast wave. Being a fast wave, it doesn’t get trapped on the surface but its energy gets leaked out. Hence, amplitude increases in +x direction and decays exponentially in the +z direction.

\[ R_n^{(2)}(x) \rightarrow j^{n+1} e^{-jx} \]

\[(4.34)\]

VI. RESULTS

Some results on dispersion characteristics are shown below. They are results for guided modes

\[
\text{dispersion characteristics of Ey mode for a/b=1 and } n_r=1.5
\]

\[
\text{Fig. 5: Dispersion characteristics of Ey mode for a/b=1 and } n_r=1.5
\]
VII. CONCLUSION

Although there are many methods to evaluate the propagation constant of a waveguide but the approximate methods are better due to their simplicity and fast convergence. They also help saving the time and calculation resources.

Marcatili’s approach is good as it is simple to implement but the circular harmonics method is better as it converges faster. Using the circular harmonics method, the dispersion characteristics for leaky modes can also be easily evaluated.

REFERENCES