Flexural Analysis of Thick T-Beam Using Fifth Order Shear Deformation Theory

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Abstract— A fifth order shear deformation theory is considered for the mathematical solution of thick beam with simply supported boundary condition. The number of variables in this theory is same as that in the first order shear deformation theory of Timoshenko. In this paper thick T-section beam is considered for to demonstrate the efficiency of theory and their results are compared with other refined shear deformation theories. This comparison shows that the present theory is more accurate than Euler-Bernoulli theory and also comparable with Timoshenko beam theory. Also, in this paper results obtained are discussed with other refined theories. The novelty of the paper is, many researchers used rectangular section for analysis, here first time irregular shape i.e. T section is considered for the purpose of analysis.

Keywords: Thick Beam, Fifth Order Shear Deformation, Equilibrium Equations, Simply Supported

I. Introduction

Since the elementary theory of beam (ETB) is based on the assumptions of Bernoulli-Euler. The elementary theory of beam (ETB) is a model of beams behave under axial forces and bending and we often used to analyze the behavior of bending elements. The elementary theory of beam (ETB) based on the major assumptions. The assumptions of elementary theory of beam (ETB) are ‘plane section remains plane’ means the section normal to neutral axis before bending remains so during bending and after bending, implying that the transverse shear strain is zero. Whereas, this assumption is not being valid to thick or deep beam. In case of thick or deep beam transverse shear deformation is predominant. It leads to less accurate result in case of isotropic thick beam. the Euler-Bernoulli beam theory (ETB) is only suitable for thin (slender) beams. This theory is almost 300 years old.

Later Timoshenko [1] developed the first order shear deformation theory (FSDT) for flexural behavior of moderately thick and thick beam in which rotary inertia and shear deformation is taken into account. Timoshenko [1] was the pioneer investigators to include refined effects such as rotary inertia and shear deformation in the beam theory. In this theory (FSDT) transverse shear strain distribution is assumed constant and requires shear correction factor. Mindlin [2] shows the Timoshenko’s shear coefficient for flexural vibration of beam. This correction factor in elementary theory of beam (ETB) and first order shear deformation theory (FSDT) led to the development of higher order or refined shear deformation theories.

The higher order shear deformation theory which are based on the trigonometric and hyperbolic functions presented by Heyliger and Reddy [3], Krishna Murty [4]. As by the higher order shear deformation theory shear stress free boundary condition is not satisfied so, this drawback removed by Ghugal and Shimpi [5] and presented review on refined shear deformation theories for isotropic and anisotropic laminated beam.

Ghugal and Sharma [6] developed a hyperbolic shear deformation theory for the static and free vibration flexural analysis of thick beam. Ghugal and Dahake [7], Dahake and Ghugal [8] employed the refined shear deformation theory for flexural of thick beam simply supported and cantilever beams. Ghumare and Sayyad [9] presented a new fifth order shear and normal deformation theory for static bending and elastic buckling of P-FGM beams. In this paper T-section beam with simply supported with uniformly varying load is considered for to demonstrate the efficiency of theories and their results are compared with other refined theories.

II. Mathematical formulation

The beam under consideration

The beam under consideration as shown in Fig. 1 occupies in $0-x-y-z$ Cartesian coordinate system the region:

$$0 \le x \le L \;;\quad 0 \le y \le b \;;\quad -h/2 \le z \le h/2 \qquad (1)$$

Where,

x, y, z = Cartesian coordinates, L = Length in x direction, B = Breadth in y directions and h = Thickness in the z-direction. The beam is made up of homogeneous, linearly elastic isotropic material.

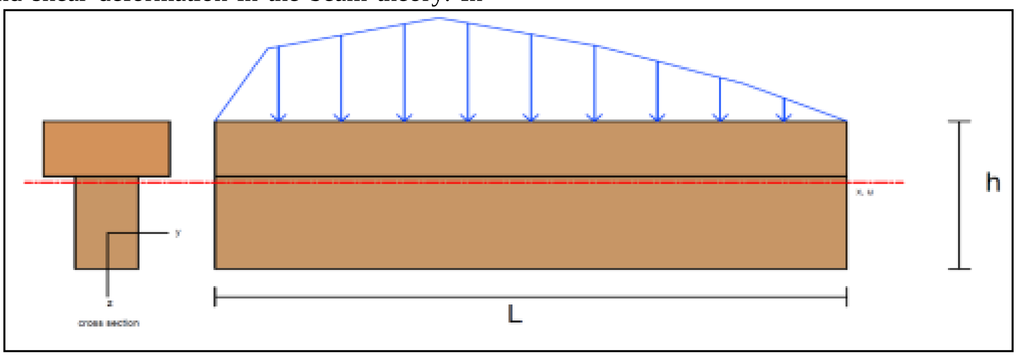

Fig. 1: Beam under bending in x-z plane

The analytical formulation of uniform thick beam based on certain kinematical and physical assumptions is presented. The variationally correct forms of differential equations and boundary conditions, based on assumed displacement field will be obtained using the principle of virtual work.

A. Strain-displacement relationships:

Normal and shear strains are obtained within the framework of linear theory of elasticity using the displacement field given by Eqn. (2). These relationships are given as follows

The Displacement Field:

$$u(x,z) = u_0 - z\frac{dw}{dx} + z\left[2 - \frac{4}{3}\left(\frac{z}{h}\right)^2 - \frac{16}{5}\frac{z^4}{h^4}\right]\phi(x) \quad (2)$$

$$w(x,\ z) = w(x)$$

Strain (Normal):

$$\varepsilon_x = \frac{du}{dx} = -z\frac{d^2w}{dx^2} + z\left(2 - \frac{4}{3}\left(\frac{z}{h}\right)^2 - \frac{16}{5}\frac{z^4}{h^4}\right)\frac{d\phi}{dx} \quad (3)$$

Strain (Shear):

$$\gamma_{zx} = \frac{du}{dz} + \frac{dw}{dx} = \left(2 - 4\frac{z^2}{h^2} - 16\frac{z^4}{h^4}\right)\phi(x) \quad (4)$$

Relationships between stress-strain:

The one dimensional Hooke's law is applied. The axial stress σ_x is related to strain ε_x and shear stress is related to shear strain by the following constitutive relations:

$$\sigma_x = E\varepsilon_x = -zE\frac{d^2w}{dx^2} + zE\left(2 - \frac{4}{3}\frac{z^2}{h^2} - \frac{16}{5}\frac{z^4}{h^4}\right)\frac{d\phi}{dx} \quad (5)$$

$$\tau_{zx} = G\gamma_{zx} = G\left(2 - 4\frac{z^2}{h^2} - 16\frac{z^4}{h^4}\right)\phi(x) \quad (6)$$

where E is modulus of elasticity and G is shear modulus of the beam material.

B. Boundary condition is Simple supports / Hinged / Pinned / Roller.

$$\frac{\partial^2 w}{\partial x^2} = \frac{d\phi}{dx} = w = 0 \ \text{……………. at } x = 0, L \quad (7)$$

III. ILLUSTRATIVE EXAMPLE

To prove the efficacy of the present theory, a numerical example is considered. The material properties for beam are as follows.

Simply supported beam (T-Section) with uniformly varying load: $q(x) = q_0\left(1 - \frac{x}{L}\right)$ = 20 kN/m

A simply supported beam has its length of 3 m and overall height of 800 mm. where T-section beam consists of flange (600 mm X 300 mm) and web (300 mm X 500 mm). The beam is subjected to uniformly varying load on surface acting in the downward *z* direction with maximum intensity of load of q_0 = 20 kN/m

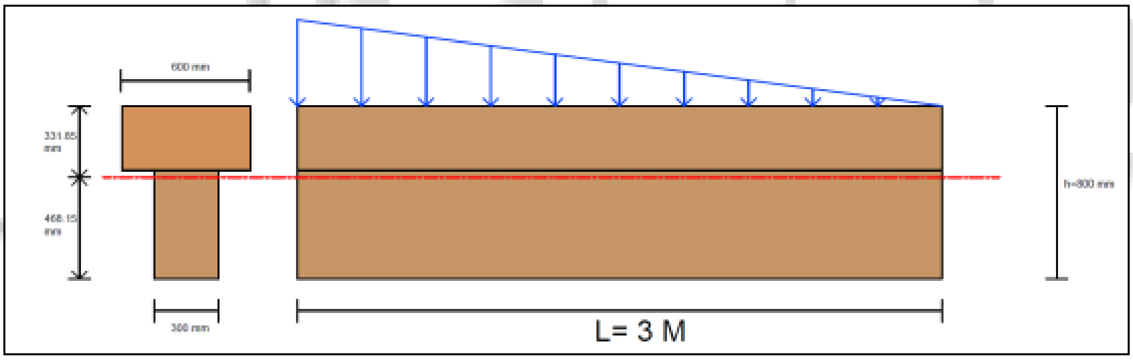

Fig. 2: T-section beam with loading condition

General expressions obtained for $w(x)$ and $\phi(x)$ are as follows:

$$w = \left\{\begin{aligned} &\frac{q_0}{GA}\frac{A_0^2 L^2}{C_0}\left(\frac{\cosh\lambda x - \sinh\lambda x - 1}{\lambda^2 L^2} - \frac{x}{(\lambda^2 L^2)L} + \frac{x}{2L} - \frac{x^2}{2L^2}\right) \\ &+ \frac{B_0}{C_0}\frac{q_0}{GA}L^2\left(\frac{x^3}{6L^3} - \frac{x}{6L}\right) \\ &- \frac{q_0}{EI}L^4\left(\frac{x^5}{120L^5} - \frac{x^4}{24L^4} + \frac{x^3}{18L^3} - \frac{x}{45L}\right) \end{aligned}\right\} \quad (8)$$

$$\phi(x) = \frac{q_0 L}{GA}\frac{A_0}{C_0}\left(\frac{\sinh\lambda x - \cosh\lambda x}{\lambda x} - \frac{x}{L} + \frac{x^2}{2L^2} + \frac{1}{3}\right) \quad (9)$$

The axial displacement and stresses obtained based on above solutions are as follows

$$u = \left\{\begin{aligned} &\frac{-z}{h}\left[\begin{aligned} &\frac{q_0}{Gb}\frac{A_0^2 L}{C_0}\left(\frac{\sinh\lambda x - \cosh\lambda x}{\lambda L} - \frac{x}{L} + \frac{1}{\lambda^2 L^2} + \frac{1}{2}\right) \\ &+ \frac{B_0}{C_0}\frac{q_0 L}{Gb}\left(\frac{x^2}{2L^2} - \frac{1}{6}\right) \\ &- \frac{12 q_0 L^3}{Ebh^2}\left(\frac{x^4}{24L^4} - \frac{x^3}{6L^3} + \frac{x^2}{6L^2} - \frac{1}{45}\right) \end{aligned}\right] \\ &+ \frac{z}{h}\left(1 - \frac{4}{3}\frac{z^2}{h^2}\right)\frac{q_0 L}{G}\frac{A_0}{C_0}\left(\frac{\sinh\lambda x - \cosh\lambda x}{\lambda L} - \frac{x}{L} + \frac{x^2}{2L^2} + \frac{1}{3}\right) \end{aligned}\right\} \quad (10)$$

$$\sigma_x = \left\{\begin{aligned} &-\frac{z}{h}\left[\begin{aligned} &\frac{q_0}{Gb}\frac{A_0^2}{C_0}(\cosh\lambda x - \sinh\lambda x - 1) + \frac{B_0}{C_0}\frac{q_0}{Gb}\left(\frac{x}{L}\right) \\ &- \frac{12 q_0 L^2}{Ebh^2}\left(\frac{x^3}{6L^3} - \frac{x^2}{2L^2} + \frac{x}{3L}\right) \end{aligned}\right] \\ &+ \frac{z}{h}\left(1 - \frac{4}{3}\frac{z^2}{h^2}\right)\frac{q_0 A_0}{GbC_0}\left(\cosh\lambda x - \sinh\lambda x - 1 + \frac{x}{L}\right) \end{aligned}\right\} \quad (11)$$

$$\tau_{zx}^{EE} = \frac{q_0}{b}\left\{ \begin{array}{l} \frac{h}{8}\left(4\frac{z^2}{h^2}-1\right)\left[\begin{array}{l} \frac{A_0^2}{GC_0L}\left(\lambda L \sinh \lambda x - \lambda L \cosh \lambda x\right) \\ +\frac{B_0}{C_0}\frac{1}{GL} - \frac{12L}{Eh^2}\left(\frac{x^2}{2L^2}-\frac{x}{L}+\frac{1}{3}\right) \end{array} \right] \\ \left[\frac{h}{8}\left(4\frac{z^2}{h^2}-1\right)-\frac{h}{48}\left(16\frac{z^4}{h^4}-1\right)\right] \\ \frac{1}{G}\frac{A_0}{C_0L}\left(\lambda L \sinh \lambda x - \lambda L \cosh \lambda x + 1\right) \end{array} \right\} \quad (12)$$

IV. RESULTS

A. Numerical Results

In this paper, the results for in plane displacement, transverse displacement, flexure and transverse shear stresses are presented. The transverse shear stresses ($\bar{\tau}_{zx}$) are obtained directly by integration of equilibrium equation of two dimensional elasticity and are denoted by ($\bar{\tau}_{zx}^{EE}$) . The transverse shear stress satisfies the stress-free boundary conditions on the top $\left(z=-h/2\right)$ and bottom $\left(z=h/2\right)$ surfaces of the beam when these stresses are obtained by both the above mentioned approaches.

Results obtained are presented in following tables

Source	Model	$\bar{w}$
Present	V Order	3.007532
Dahake and Ghugal	TSDT	3.006984
Ghugal and Sharma	HPSDT	3.009138
Krishna Murty	HSDT	3.008396
Timoshenko	FSDT	2.687999
Bernoulli-Euler	ETB	2.687998

Table 1: Transverse Deflection ($\bar{w}$) of the simply supported Beam Subjected T-beam

Source	At	Model	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{EE}$
Present	Above NA	V Order	-0.34380	0.00077	0.00001088
Dahake and Ghugal		TSDT	-0.33139	0.00073	0.00000925
Ghugal and Sharma		HPSDT	-0.18655	0.00015	-0.00000977
Krishna Murty		HSDT	-0.33226	0.00073	0.00000934
Timoshenko		FSDT	-0.20695	0.00073	0.00001533
Bernoulli-Euler		ETB	-0.27708	0.00089	0.00003833
Present	At NA	V Order	0	0	0.00007249
Dahake and Ghugal		TSDT	0	0	0.00005961
Ghugal and Sharma		HPSDT	0	0	-0.00009090
Krishna Murty		HSDT	0	0	0.00006036
Timoshenko		FSDT	0	0	0.00010416
Bernoulli-Euler		ETB	0	0	0.00026041
Present	Below NA	V Order	0.95712	-0.00218	-0.00002601
Dahake and Ghugal		TSDT	0.93229	-0.00208	-0.00002273
Ghugal and Sharma		HPSDT	0.64410	-0.00093	+0.00001527
Krishna Murty		HSDT	0.93418	-0.00209	-0.00002293
Timoshenko		FSDT	0.57158	-0.00202	-0.00003600
Bernoulli-Euler		ETB	0.76528	-0.00246	-0.00009000

Table 2. The results for axial displacement (u), axial stress ($\boldsymbol{\sigma}_x$), transverse shear stress τ_{zx}^{EE} of simply supported T-beam

B. Graphical Results

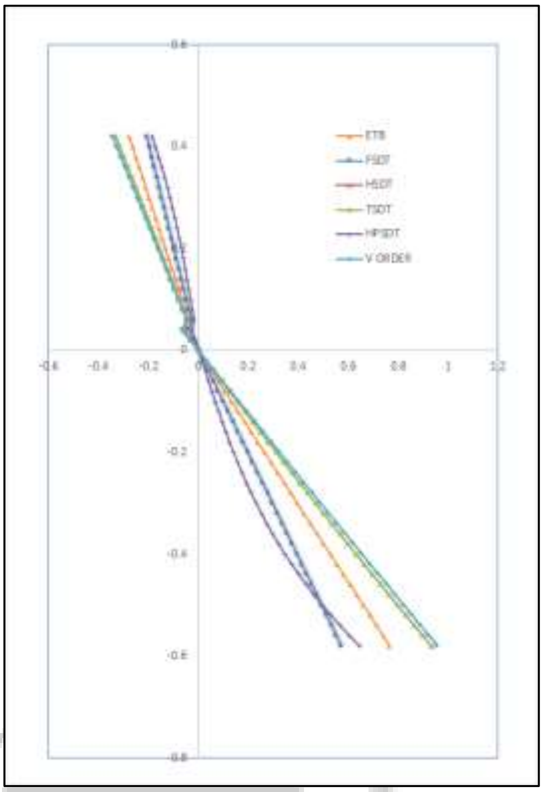

Fig. 3: Variation of axial displacement ($\bar{u}$) through the thickness of simply supported beam from top $+h/2$ to bottom $-h/2$ when subjected to uniformly varying load.

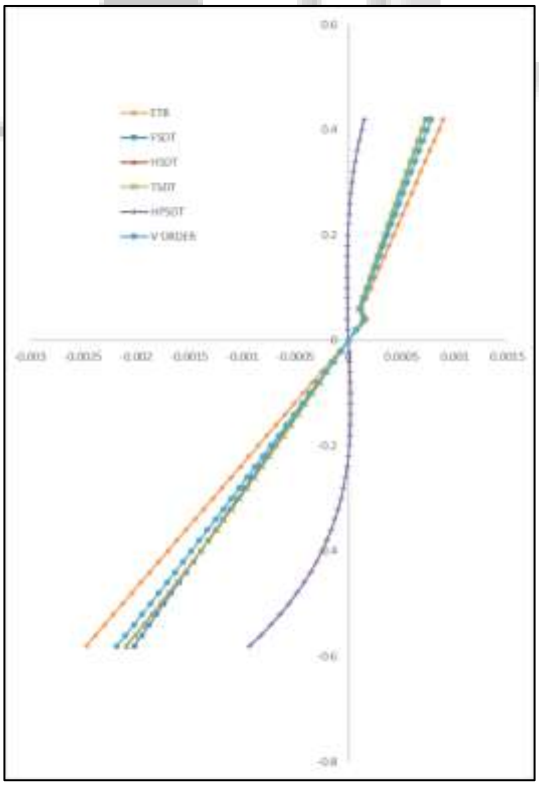

Fig. 4: Variation of axial stress ($\bar{\sigma}_x$) through the thickness of simply supported beam from top $+h/2$ to bottom $-h/2$ when subjected to uniformly varying load.

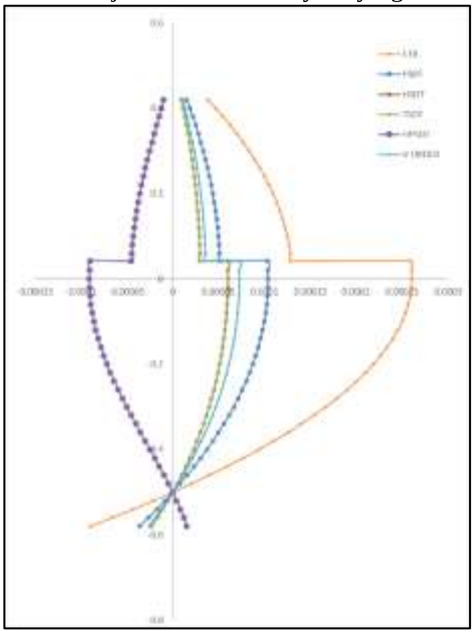

Fig. 5: Variation of transverse shear stress ($\bar{\tau}_{zx}$) through the thickness of simply supported beam from top $+h/2$ to bottom $-h/2$ when subjected to uniformly varying load.

C. Discussion of Results

The results obtained from the present fifth order shear deformation theory (V Order) are compared with those of the elementary beam theory (ETB/ Classical), first order shear deformation theory (FSDT) of Timoshenko, higher order shear deformation theory of Krishna Murty and hyperbolic shear deformation theory (HPSDT) of Ghugal and Sharma. The results in this section are discussed as follows.

The comparison of results of Variation of axial displacement ($\bar{u}$) is presented in Table 2 through the thickness of simply supported beam from top $+h/2$ to bottom $-h/2$ when subjected to uniformly varying load (see fig.3). The results of axial displacement given by present theory are in close relation with the results of other refined theories. Through the thickness of beam the distribution of axial displacement ($\bar{u}$) is in close relation with other refined theories except hyperbolic shear deformation theory (HPSDT) of Ghugal and Sharma and first order shear deformation theory (FSDT) of Timoshenko. The displacement component of hyperbolic shear deformation theory (HPSDT) of Ghugal and Sharma through the thickness of beam from from top $+h/2$ to bottom $-h/2$ shows parabolic variation and the displacement component of first order shear deformation theory (FSDT) of Timoshenko through the thickness of beam from from top $+h/2$ to bottom

$-h/2$ shows straight line which is not in close relation with other refined theories as shown in Fig. 3.

The comparison of results of Variation of axial stress ($\bar{\sigma}_x$) is presented in Table no.2 through the thickness of simply supported beam from top $+h/2$ to bottom $-h/2$ when subjected to uniformly varying load (see fig.4). The results of axial stress given by present theory are in close relation with the results of other refined theories. Through the thickness of beam the distribution of axial stress ($\bar{\sigma}_x$) is in close relation with other refined theories except hyperbolic shear deformation theory (HPSDT) of Ghugal and Sharma. The axial stress obtained from HPSDT shows non-linear variation through the thickness of beam as shown in Fig. 4

V. Conclusions

A fifth order refined shear deformation theory for flexure of T-section thick beam with simply support condition with uniformly varying load is presented. The results obtained are discussed with those of other refined theories. From the results and discussion of present study following conclusions are drawn.

1) The transverse displacements, axial stresses and their distribution, transverse shear stresses obtained from constitutive relation are in close agreement with the other shear deformation theories.
2) It shows that the results obtained from hyperbolic shear deformation theory (HPSDT) of Ghugal and Sharma are not in relation other refined theories.
3) In case of unsymmetric thick beam, for transverse shear stresses at top and bottom does not show zero stresses.

In general, the use of present theory gives accurate results in a simple way as seen from the numerical example studied and it can predict the local effects in the vicinity of the built-in end of the thick beams. This validates the efficacy and credibility of fifth order shear deformation theory.

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